Sinyaller ve Sistemler

“Fourier Dönüşümü”

Dr. Cahit Karakuş, 2020

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

$$\int \frac{1}{x} \, dx = \ln x + c$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

$$\int \ln x \, dx = x(\ln x - 1) + c$$

$$\int x \, e^{ax} \, dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right) + c$$

$$\int \sin x \, dx = -\cos x + c$$

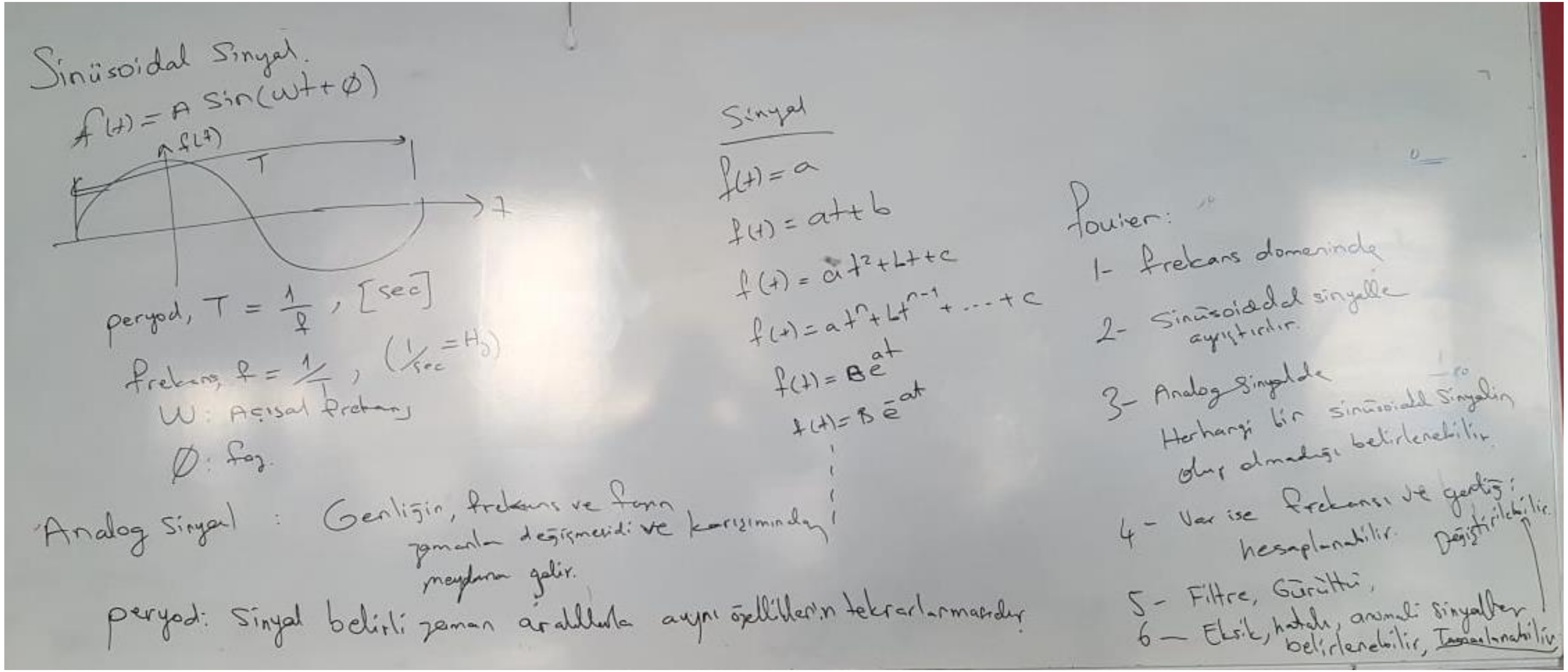
$$\int \cos x \, dx = \sin x + c$$

1. $\frac{d}{dx}(c) = 0,$ where c is a constant
2. $\frac{d}{dx}(x^n) = nx^{n-1},$ where n is any real number
3. $\frac{d}{dx}(e^x) = e^x$
4. $\frac{d}{dx}(\ln x) = \frac{1}{x},$ for $x>0$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\cos x) = -\sin x$

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

$$(uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Sinüsoidal Sinyal



Fourier Theory

Fourier Theory

- Bir Fransız matematikçi ve fizikçi Jean Baptiste Joseph Fourier, Fourier analizini geliştirdi. Periyodik sinyalin (Analog) uygun seçilmiş sinüzoidal dalgaların toplamı olarak temsil edilebileceği konusunda tartışmalı bir iddiaya sahipti. Bu yazının bir gözden geçiricisi olan matematikçi Lagrange, süreksiz eğimler gibi köşeleri olan sinyalleri temsil etmek için bir yaklaşımın kullanılamayacağı konusunda ısrar etti. Lagrange'ın görüşü doğruydu ama tam olarak değil, çünkü sıfır enerjiye sahip iki sinüzoidal işaret arasındaki fark çok yakındı. Makale sonunda Lagrange öldükten sonra yayınlandı. Fourier'in genelleme iddiasının biraz kuvvetli olduğu ortaya çıksa da, sonuçları günümüze kadar devam eden önemli bir araştırma selini harekete geçirdi.

Jean Baptiste Joseph Fourier (1768-1830)

- Had crazy idea (1807):
- Herhangi bir periyodik fonksiyon (Analog Sinyal), farklı frekanslardaki sinüs ve kosinüslerin yani sinüsoidal sinyallerin ağırlıklı toplamı olarak yeniden yazılabilir.
- Don’t believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it’s true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

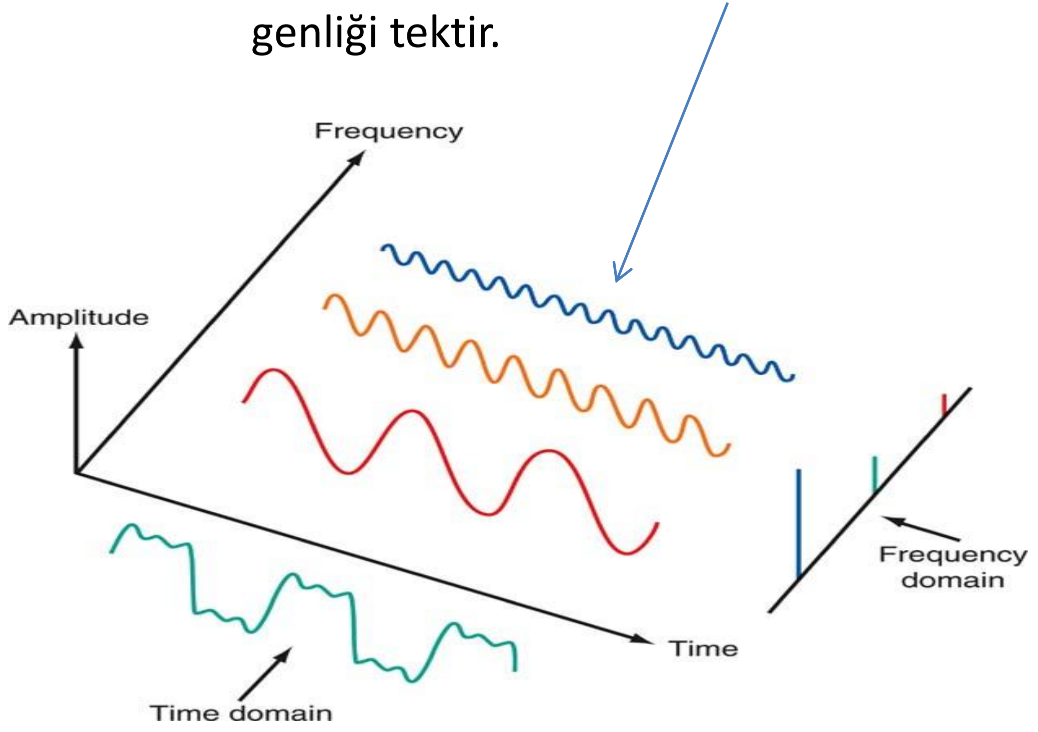
Fourier Theory

- Özellikle sinüs dışı dalga yaklaşımı için bir iletişim devresine veya sistemine ait sinyallerin karakteristiklerini ve performansını belirlemek için kullanılan bir yöntem Fourier analizidir.
- Fourier teorisi, sinüzoidal olmayan sinyalde bir dalga formunun, harmonik olarak ilişkili bireysel sinüs dalgası veya kosinüs dalgası (sinüsoidal sinyal) bileşenlerine ayrılabileceğini belirtir.
- Bir kare dalga bu fenomenin klasik bir örneğidir.

Temel İçerik:

- Fourier analizi, bir sinyal sonsuz sayıda harmonik sinüsoidal sinyallerden oluştuğunu belirtir.
- Fourier analizi, yalnızca karmaşık bir sinyaldeki sinüs dalgası bileşenlerini (frekans, genlik, faz) değil, aynı zamanda bir sinyalin bant genişliğini belirlememizi sağlar.

Sinüsoidal (Analog) sinyalin frekansı, fazı, genliği tektir.



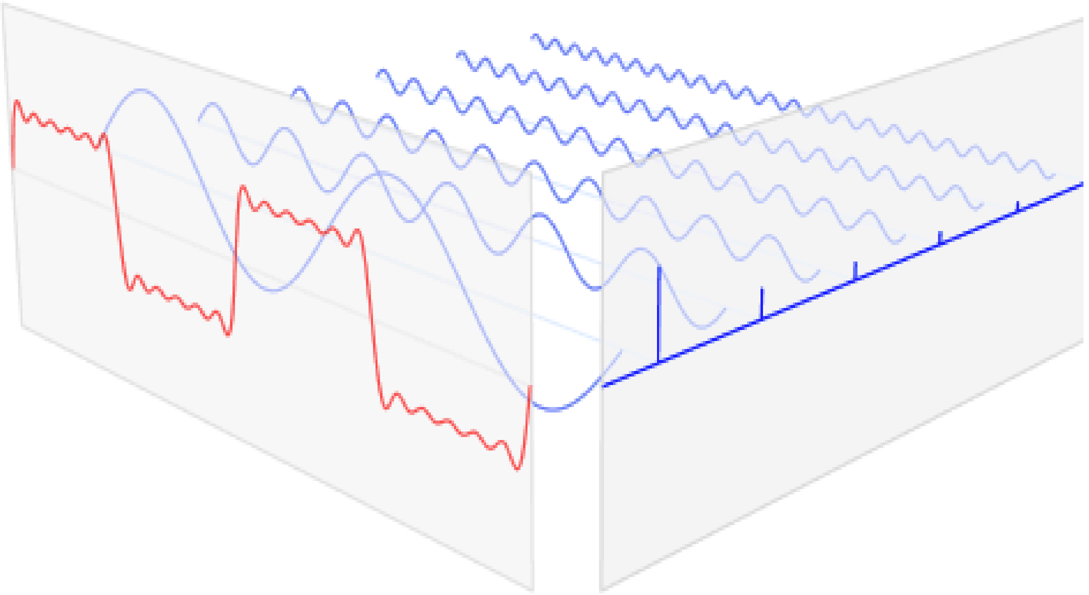
Fourier Theory

Zaman Etki Alanı (domeni) ve Frekans Etki Alanı (domeni)

- Gerilim, akım veya güç gibi sinyallerin değişimlerinin zamana göre genliğinin nasıl değiştiği zaman domeninde ifade edilir.
- Bir frekans domeni, sinyali oluşturan frekanslara göre genlik değişimlerini çizer.
- Fourier teorisi bize karmaşık sinyalleri, frekans açısından ifade etmek ve göstermek için yeni ve farklı bir yol sunar.
- Zaman domenine karşı frekans domeni: Spektrum analizörü, sinyalin bir frekans alanına ait ekranı oluşturmak için kullanılan bir araçtır. İletişim ekipmanının tasarlanması, analizi ve sorunlarının giderilmesinde anahtar test aracıdır.

Fourier dönüşümü

- Fourier dönüşümü yaygın olarak zaman spektrumundaki bir sinyali bir frekans spektrumuna dönüştürmek için kullanılır. Zaman spektrumlarına sinyal örnekleri olarak ses dalgaları, elektriksel sinyaller, mekanik titreşimler vb. Aşağıdaki şekil açıkça görülebileceği gibi, Kare dalga farklı frekanslara sahip bir dalgaya benziyor. Aslında birden çok dalgaya benziyor.
- Fourier Dönüşümü doğrusal olmayan her fonksiyonun (sonsuz) sinüs dalgalarının bir toplamı olarak temsil edilebileceği gerçeğinden yararlanır. Alttaki şekilde bu, bir basamak fonksiyonu çok sayıda sinüs dalgası tarafından simüle edildiği için gösterilmiştir.



Fourier Dönüşümü

- Evrende gözlediğiniz tüm sinyal formları farklı frekans ve genliklere sahip sinüs fonksiyonlarının toplamından ibarettir!
- Sinyaller, bir dalga formu aracılığıyla tanımlanabilir - zaman, mekan veya başka bir değişkenin fonksiyonu. Örneğin, ses dalgaları, elektromanyetik alanlar, radyodan dinlediğiniz müzik, hisse senetlerini zamana göre fiyatı, nefesinizin sıklığı vb.
- Fourier Dönüşümü, bize bu dalga formlarını doğrudan görüntülemenin benzersiz ve güçlü bir yolunu sunduğu için önemli bir rol oynamaktadır.
- Fourier Transform, sinyal analizi, görüntü analizi, görüntü filtreleme, görüntü rekonstrüksiyonu ve görüntü sıkıştırması gibi çok çeşitli uygulamalarda kullanılır.
- Fourier dönüşümü fizik ve mühendislik alanındaki birçok uygulama ile matematiksel bir dönüşümdür. Fourier serileri denilen trigonometrik serileri kullanarak kısmi diferansiyel denklemleri içeren birçok önemli problemi çözebiliriz.

Fourier Transform

- Fourier dönüşümü (Fourier Transform) sıklık (frekans)analizinde kullanılan, istatistik tabanlı, matematiksel bir işlemdir.

- Zaman domenindeki karışık sinyal yumaklarını ayrıştırır ve hangi frekansta ne şiddette (genlik) bir sıklık olduğunu gösterir. Kısaca sinyallerimizi zaman alanından frekans alanına geçirirken kullandığımız bir işlemdir.

- Fourier dönüşümü peryodik olarak tekrarlanmayan sinyalleri dikkate almaz. Karmaşık sinyaller içinde periyodik olanları belirleyip harmonik bileşenlerine ayırır.

Fourier Transform

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad \text{Analysis}$$

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \qquad \text{Synthesis}$$

Continuous-Time Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega) e^{j\omega t} d\omega$$

Discrete-Time Fourier Transform(DTFT):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} X(e^{j\omega}) e^{-j\omega n}$$

- Zamanla genliği değişen analog sinyallerindeki frekansların belirlenmeside ve frekansa göre genliklerinin ayrıştırma Fourier dönüşümü.
- Frekansların bileşenlerinden zaman domeninde analog sinyal elde edilmesi sentez. Frekanslara göre birleştirme.
- Peryodik Sinyallarin Fourier Dönüşümü daha doğru hesaplanır.
- Bir analog sinyalin Fourier dönüşümü komplks sayılar içerir.

Discrete Fourier Transform

A Fourier Transform will break apart a time signal and will return information about the frequency of all sine waves needed to simulate that time signal. For sequences of evenly spaced values the Discrete Fourier Transform (DFT) is defined as:

$$X_k = \sum_{n=0}^{N-1} x_n e^{-2\pi ikn/N}$$

Where:

- N = number of samples
- n = current sample
- x_n = value of the signal at time n
- k = current frequency (0 Hz to N-1 Hz)
- X_k = Result of the DFT (amplitude and phase)

Properties of Fourier Transform

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi ux}$)

Fourier Series Cosine-Sine Form

$$x(t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\omega_1 t + B_n \sin n\omega_1 t)$$

$$\omega_1 = 2\pi f_1 = \frac{2\pi}{T}$$

$$f_1 = \frac{1}{T}$$

Açısal frekans dalga biçiminde hareket eden sinyallerde söz konusudur.

The Fourier series of the function $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\},$$

where the Fourier coefficients a_0, a_n, and b_n are defined by the integrals

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.$$

Discrete Fourier Series Complex Exponential Form

$$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{X_n} e^{in\omega_1 t}$$

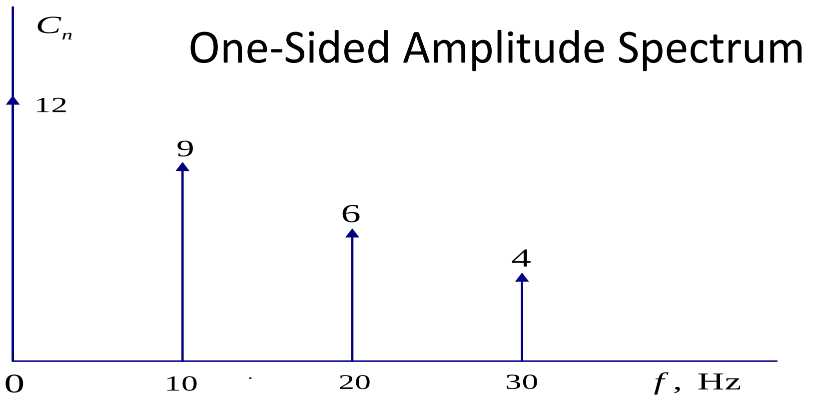
$$\mathbf{X_n} = \frac{1}{T}\int_0^T x(t) e^{-in\omega_1 t} dt$$

Example: frequencies and plot the one-sided amplitude spectrum.

- X(t)=Acos(wt+θ), x(t): Analog sinyal, eğer frekans tek ise sinüsoidal sinyal olarak adlandırılır.
- A: Genlik
- w: Açısal frekans (Dalga biçiminde yayılan bir sinyal söz konusu)
- w=2πf , f=1/T; Burada f: frekans (Hz), T: peryod (saniye)
- Θ: faz açılarıdır, radyan cinsinden verilir (derecede verilebilir).

The frequencies are
0 (dc)
10 Hz
20 Hz
30 Hz

$$x(t) = 12 + 9\cos(2\pi \times 10t + \pi/3) + 6\cos(2\pi \times 20t - \pi/6) + 4\cos(2\pi \times 30t + \pi/4)$$



Örnek

- 4 adet sinüsoidal sinyalin toplamından oluşan analog sinyali elde ediniz. (π rad=180 derece)
- A0=12 birim, A1=9birim, A2=6birim, A3=4 Birim
- Frekans: fo=0Hz, f1= 10Hz, f2=20Hz, f3=30Hz
- Faz açıları: fi0=0 derece, fi1= 60derece, fi2=-30 derece, fi3=45 derece

$$
\begin{aligned}
x(t) = 12 + 9\cos(2\pi \times 10t + \pi / 3) \\
+ 6\cos(2\pi \times 20t - \pi / 6) \\
+ 4\cos(2\pi \times 30t + \pi / 4)
\end{aligned}
$$

```
clear all
close all
f1=10
f2=20
df=1;
Fs=f2*5;
Ts=1/5;
dt=Ts/100
f=0:df:Fs;
t=0:dt:Ts
NA=size(f)
y=cos(2*pi*f1*t) +5*sin(2*pi*f2*t)+2*rand(size(t));
figure, plot(t,y)
grid on
fa=fft(y);
fb=fftshift(fa);
figure, plot(f,abs(fa))

N1=length(fa)
```

```
for i=1:N1
   fc(i)=0;
   if i>=95,
      fc(i)=fa(i);
   end
end
  figure, plot(abs(fc))

ft=ifft(fc);
figure, plot(real(ft))
grid on
```

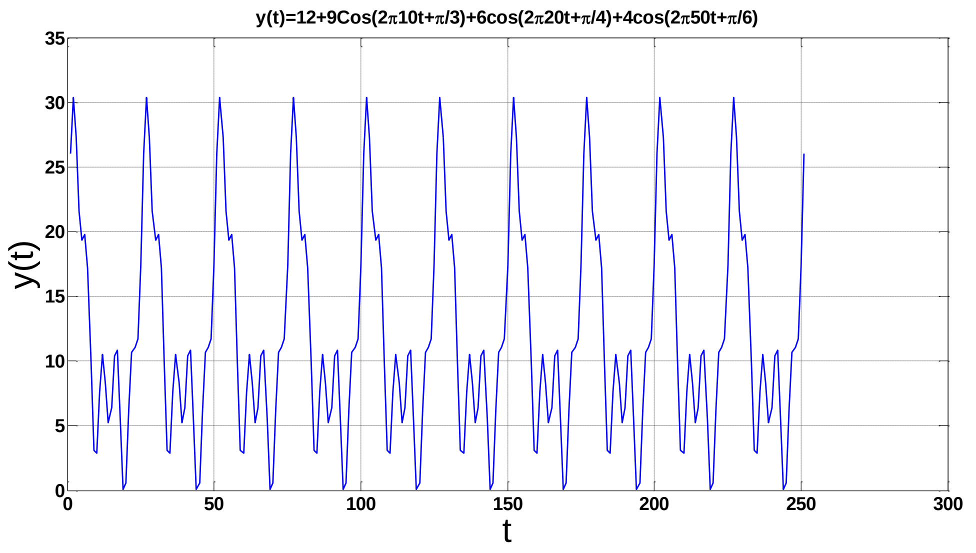
Frekanslar belirlenir.
Fs>=2*fmaks alınır.
Fs=20*10=200, Ts=1/200

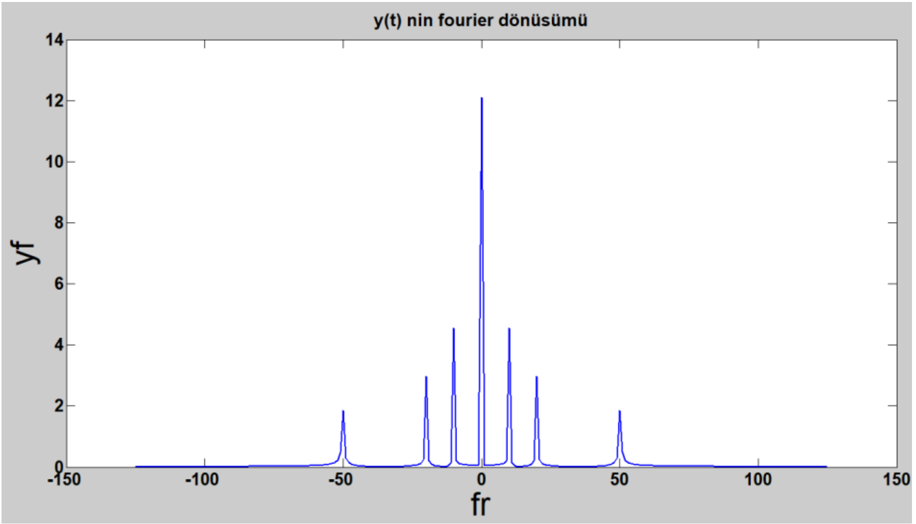
Örnek

```
clear all;  close all
f1=0; f2=10;f3=20;f4=50;
A1=12;,A2=9;A3=6;, A4=4;
faz1=0; faz2=pi/3;faz3=pi/4;faz4=pi/6;

fs=5*f4
p=fs/f2
Ts=1/fs
N=round(1/Ts)

for i=1:N+1
   t(i)=0+(i-1)*Ts;
   fr(i)=-round(N/2)+i-1;
end

for i=1:N+1
   y1(i)=A1;
   y2(i)=A2*sin(2*pi*f2*t(i)+pi/3);
   y3(i)=A3*sin(2*pi*f3*t(i)+pi/4);
   y4(i)=A4*sin(2*pi*f4*t(i)+pi/6);
   y(i)=y1(i)+y2(i)+y3(i)+y4(i);
end
```

```
figure, plot(y,'LineWidth', 2, 'MarkerSize', 10)
grid on;
title('y(t)=12+9Cos(2\pi10t+\pi/3)+6cos(2\pi20t+\pi/4)+4cos(2\pi50t+\pi/6)', 'FontSize', 20, 'Color', 'k', 'FontWeight', 'bold');
xlabel('t','FontSize',36)
ylabel('y(t)','FontSize',36)
set(gca,'FontSize',20, 'FontWeight','bold');
```

```
S0=fft(y);
S1=abs(S0)/N;
figure, plot(fr,fftshift(S1),'LineWidth', 2, 'MarkerSize', 10)
title('y(t) nin fourier dönüsümü', 'FontSize', 20, 'Color', 'k', 'FontWeight', 'bold');
xlabel('fr','FontSize',36)
ylabel('yf','FontSize',36)
set(gca,'FontSize',20, 'FontWeight','bold');
```





Example 2.

Find the Fourier series for the square 2π-periodic wave defined on the interval $[-\pi, \pi]$:

$$f(x) = \begin{cases} 0, & \text{if} \quad -\pi \le x \le 0 \\ 1, & \text{if} \quad 0 < x \le \pi \end{cases}.$$

Solution.

First we calculate the constant a_0 :

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{0}^{\pi} 1 dx = \frac{1}{\pi} \cdot \pi = 1.$$

Find now the Fourier coefficients for $n \neq 0$:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{0}^{\pi} 1 \cdot \cos nx dx = \frac{1}{\pi} \left[\left(\frac{\sin nx}{n} \right) \Big|_0^{\pi} \right] = \frac{1}{\pi n} \cdot 0 = 0,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{0}^{\pi} 1 \cdot \sin nx dx = \frac{1}{\pi} \left[\left(-\frac{\cos nx}{n} \right) \Big|_0^{\pi} \right]$$

$$= -\frac{1}{\pi n} \cdot (\cos n\pi - \cos 0) = \frac{1 - \cos n\pi}{\pi n}.$$

```
clear all;  close all
syms t n
n=1;
f=1;
f1=f*cos(n*t);
f2=f*sin(n*t);
a0=int(f,t,0,pi)/pi
a1=int(f1,t,0,pi)/pi
b1=int(f2,t,0,pi)/pi
```

```
a0 = 1
a1 = 0
b1 = 2/pi
```

Example

Let $f(x)$ be a 2π-periodic function such that $f(x) = x^2$ for $x \in [-\pi, \pi]$. Find the Fourier series for the parabolic wave.

Solution.

Since this function is even, the coefficients $b_n = 0$. Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx = \frac{2}{\pi}\cdot\left[\left(\frac{x^3}{3}\right)\Big|_0^{\pi}\right] = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx.$$

```
clear all;  close all
syms t n
n=1
f=t^2;
f1=f*cos(n*t);
f2=f*sin(n*t);
a0=int(f,t,-pi,pi)/pi
a1=int(f1,t,-pi,pi)/pi
b1=int(f2,t,-pi,pi)/pi

n =   1
a0 =(2*pi^2)/3
a1 = -4
b1 = 0
```

Apply integration by parts twice to find:

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx = \begin{bmatrix} u = x^2 \\ dv = \cos nx dx \\ du = 2x dx \\ v = \int \cos nx dx = \frac{\sin nx}{n} \end{bmatrix} = \frac{2}{\pi}\Big[\left(\frac{x^2 \sin nx}{n}\right)\Big|_0^{\pi}$$

$$- \int_0^{\pi} 2x \frac{\sin nx}{n} dx\Big] = \frac{2}{\pi n}\left[\pi^2 \sin n\pi - (-\pi)^2 \sin(-n\pi) - 2\int_0^{\pi} x \sin nx dx\right]$$

$$= \frac{2}{\pi n}\left[2\pi^2 \sin n\pi - 2\int_0^{\pi} x \sin nx dx\right] = -\frac{4}{\pi n}\int_0^{\pi} x \sin nx dx$$

$$= \begin{bmatrix} u = x \\ dv = \sin nx dx \\ du = dx \\ v = \int \sin nx dx = -\frac{\cos nx}{n} \end{bmatrix} = -\frac{4}{\pi n}\left[\left(-\frac{x \cos nx}{n}\right)\Big|_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos nx}{n}\right) dx\right]$$

$$= \frac{4}{\pi n^2}\left[\pi \cos n\pi - \int_0^{\pi} \cos nx dx\right] = \frac{4}{\pi n^2}\left[\pi \cos n\pi - \left(\frac{\sin nx}{n}\right)\Big|_0^{\pi}\right]$$

$$= \frac{4}{\pi n^2}\left[\pi \cos n\pi - \frac{\sin n\pi}{n}\right].$$

Example

Find the Fourier series for the triangle wave

$$f(x) = \begin{cases} \frac{\pi}{2} + x, & \text{if} \quad -\pi \le x \le 0 \\ \frac{\pi}{2} - x, & \text{if} \quad 0 < x \le \pi \end{cases},$$

defined on the interval $[-\pi, \pi]$.

Solution.

The constant a_0 is

$$a_0 = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \left[\int\limits_{-\pi}^{0} \left(\frac{\pi}{2} + x\right) dx + \int\limits_{0}^{\pi} \left(\frac{\pi}{2} - x\right) dx \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{\pi}{2}x + \frac{x^2}{2} \right)\Big|_{-\pi}^{0} + \left(\frac{\pi}{2}x - \frac{x^2}{2} \right)\Big|_{0}^{\pi} \right] = \frac{1}{\pi} \left[0 - \left(-\cancel{\frac{\pi^2}{2}} + \cancel{\frac{(-\pi)^2}{2}} \right) \right.$$

$$\left. + \left(\cancel{\frac{\pi^2}{2}} - \cancel{\frac{\pi^2}{2}} \right) - 0 \right] = 0.$$

$$a_n = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left[\int\limits_{-\pi}^{0} \left(\frac{\pi}{2} + x \right) \cos nx dx + \int\limits_{0}^{\pi} \left(\frac{\pi}{2} - x \right) \cos nx dx \right]$$
$$= \frac{1}{\pi} \left[\int\limits_{-\pi}^{0} \frac{\pi}{2} \cos nx dx + \int\limits_{-\pi}^{0} x \cos nx dx + \int\limits_{0}^{\pi} \frac{\pi}{2} \cos nx dx - \int\limits_{0}^{\pi} x \cos nx dx \right].$$

Integrating by parts, we can write

$$\int x \cos nx dx = \frac{x \sin nx}{n} - \int \frac{x \sin nx}{n} dx = \frac{x \sin nx}{n} + \frac{\cos nx}{n^2}.$$

Then

$$a_n = \frac{1}{\pi} \left[\frac{\pi}{2} \left(\frac{\sin nx}{n} \right) \Big|_{-\pi}^{0} + \left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right) \Big|_{-\pi}^{0} + \frac{\pi}{2} \left(\frac{\sin nx}{n} \right) \Big|_{0}^{\pi} \right.$$
$$\left. - \left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right) \Big|_{0}^{\pi} \right].$$

The values of $\sin nx$ at $x = 0$ or $x = \pm\pi$ are zero. Therefore,

$$a_n = \frac{1}{\pi n^2} \left[(\cos nx)|_{-\pi}^{0} - (\cos nx)|_{0}^{\pi} \right] = \frac{1}{\pi n^2} [\cos 0 - \cos(-\pi n) - \cos \pi n + \cos 0]$$
$$= \frac{2}{\pi n^2} [1 - \cos \pi n] = \frac{2}{\pi n^2} [1 - (-1)^n].$$

Example

Find the Fourier series for the function

$$f(x) = \begin{cases} -1, & \text{if } -\pi \le x \le -\frac{\pi}{2} \\ 0, & \text{if } -\frac{\pi}{2} < x \le \frac{\pi}{2} \\ 1, & \text{if } \frac{\pi}{2} < x \le \pi \end{cases},$$

defined on the interval $[-\pi, \pi]$.

Solution.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} (-1)\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0dx + \int_{\frac{\pi}{2}}^{\pi} 1dx\right]$$

$$= \frac{1}{\pi}\left(-\frac{\cancel{\pi}}{\cancel{2}} + 0 + \frac{\cancel{\pi}}{\cancel{2}}\right) = 0.$$

Compute the coefficients a_n :

$$a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int\limits_{-\pi}^{-\frac{\pi}{2}} (-\cos nx)\, dx + \int\limits_{\frac{\pi}{2}}^{\pi} \cos nx dx\right]$$

$$= \frac{1}{\pi n}\left[-\sin\left(-\frac{n\pi}{2}\right) + \sin(-n\pi) + \sin n\pi - \sin\frac{n\pi}{2}\right] = \frac{1}{\pi n}\left[\cancel{\sin\frac{n\pi}{2}} - \cancel{\sin n\pi}\right.$$

$$\left. + \cancel{\sin n\pi} - \cancel{\sin\frac{n\pi}{2}}\right] = 0.$$

(These results are obvious since this function is odd.)

Calculate the coefficients b_n :

$$b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left[\int\limits_{-\pi}^{-\frac{\pi}{2}} (-\sin nx)\, dx + \int\limits_{\frac{\pi}{2}}^{\pi} \sin nx dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n}\right)\Big|_{-\pi}^{-\frac{\pi}{2}} - \left(\frac{\cos nx}{n}\right)\Big|_{\frac{\pi}{2}}^{\pi}\right] = \frac{1}{\pi n}\left[\cos\left(-\frac{n\pi}{2}\right) - \cos(-n\pi)\right.$$

$$\left. - \cos n\pi + \cos\frac{n\pi}{2}\right] = \frac{1}{\pi n}\left[\cos\frac{n\pi}{2} - \cos n\pi - \cos n\pi + \cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi n}\left(\cos\frac{n\pi}{2} - \cos n\pi\right).$$

Thus, the Fourier series expansion of the function is given by

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin nx.$$

Example

$$f(t) = e^{i\omega_o t}$$

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega_o t}e^{-i\omega t}\,dt$$

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-\omega_o)t}\,dt$$

$$g(\omega) = \frac{1}{2\pi}(2\pi\delta(\omega - \omega_o))$$

$$g(\omega) = \delta(\omega - \omega_o)$$

$$g(\omega) = \delta(\omega - \omega_o)$$

$$f(t) = \int_{-\infty}^{\infty} g(\omega)e^{i\omega t}\,d\omega$$

$$f(t) = \int_{-\infty}^{\infty} \delta(\omega - \omega_o)e^{i\omega t}\,d\omega$$

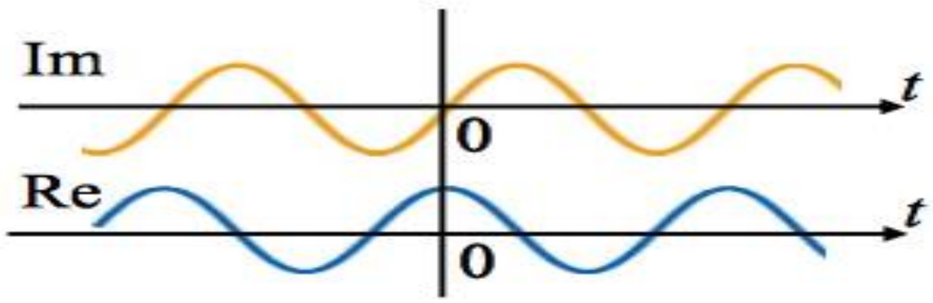
$$f(t) = \begin{cases} e^{i\omega_o t} & \omega = \omega_o \\ 0 & \omega \neq \omega_o \end{cases}$$

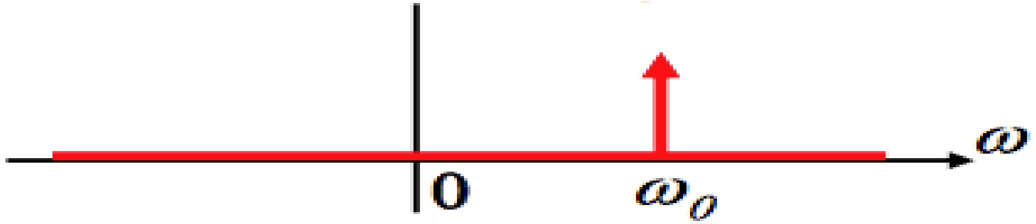
$$f(t) = e^{i\omega_o t}$$

Example cont.

Fourier Transform

$f(t) = e^{i\omega_o t}$ $\longrightarrow$ $g(\omega) = \delta(\omega - \omega_o)$





$\longleftarrow$

Fourier Inverse Transform

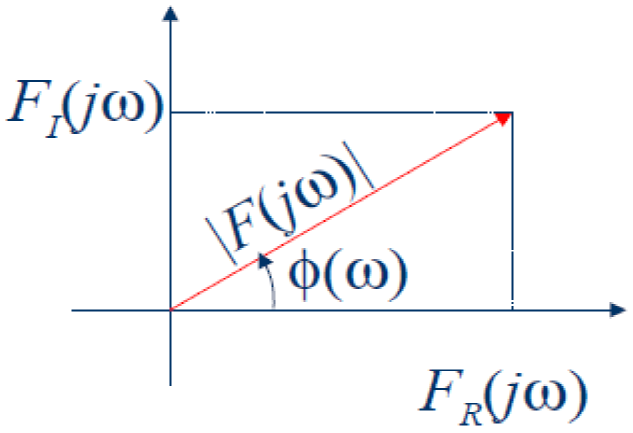
Continuous Spectra

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$F(j\omega) = F_R(j\omega) + jF_I(j\omega)$$

$$= |F(j\omega)|e^{j\phi(\omega)}$$

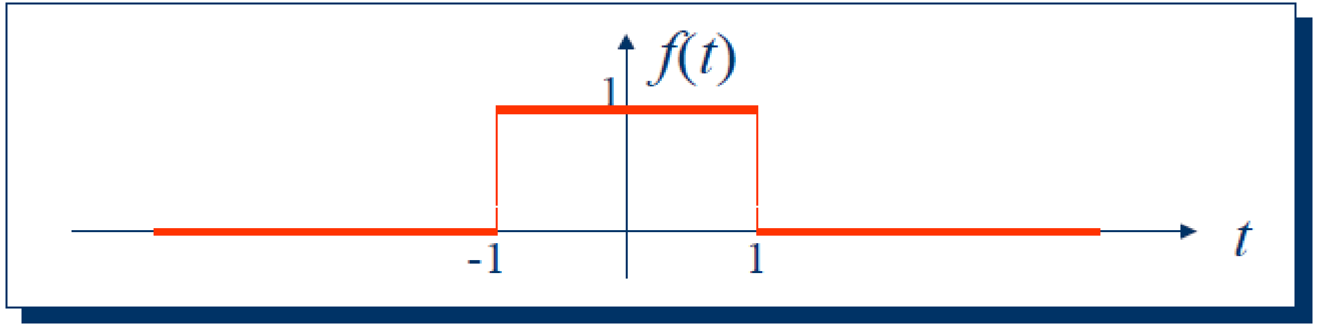
Magnitude: $|F(j\omega)|$ — Phase: $\phi(\omega)$



Example

Fourier Transform:

$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

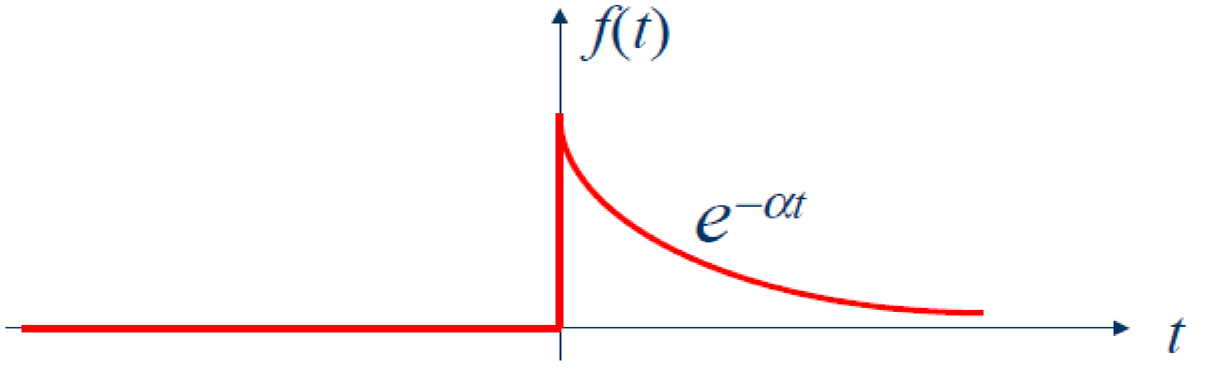


$$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt=\int_{-1}^{1} e^{-j\omega t}dt=\frac{1}{-j\omega}e^{-j\omega t}\Big|_{-1}^{1}$$

$$=\frac{j}{\omega}(e^{-j\omega}-e^{j\omega})=\frac{2\sin\omega}{\omega}$$

Example: Determine the Fourier transform of the function below.

$$x(t) = e^{-\alpha t} \quad \text{for} \quad t \geq 0$$
$$= 0 \quad \text{for} \quad t < 0$$



$$\mathbf{X}(f) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$
$$= \int_0^{\infty} e^{-\alpha t} e^{-i\omega t} dt = \int_0^{\infty} e^{-(\alpha + i\omega)t} dt$$

$$\mathbf{X}(f) = \left. \frac{e^{-(\alpha+i\omega)t}}{-(\alpha+i\omega)} \right]_0^{\infty}$$
$$= 0 - \frac{1}{-(\alpha+i\omega)} = \frac{1}{(\alpha+i\omega)}$$

$$X(f) = |\mathbf{X}(f)| = \left| \frac{1}{\alpha + i\omega} \right| = \frac{1}{\sqrt{\alpha^2 + \omega^2}} \qquad \theta(f) = -\tan^{-1} \frac{\omega}{\alpha}$$

Example: Determine the Fourier transform of the function below.

$$x(t) = A \quad \text{for} \quad 0 < t < \tau$$
$$= 0 \qquad \text{elsewhere}$$

$$\mathbf{X}(f) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt = \int_0^{\tau} Ae^{-i\omega t}\,dt = \left.\frac{Ae^{-i\omega t}}{-i\omega}\right]_0^{\tau}$$

$$= A\left(\frac{e^{-i\omega\tau} - 1}{-i\omega}\right) = A\left(\frac{1 - e^{-i\omega\tau}}{i\omega}\right)$$

$$\mathbf{X}(f) = A\tau\left(\frac{\sin \pi f \tau}{\pi f \tau}\right)e^{-i\pi f \tau} \qquad X(f) = A\tau\left(\frac{\sin \pi f \tau}{\pi f \tau}\right)$$

$$\theta(f) = -\pi\tau f$$

Example:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$\mathcal{F}[f(t)] = F(j\omega)$ $\qquad \mathcal{F}[f(t)\cos\omega_0 t] = ?$

$$f(t)\cos\omega_0 t = \frac{1}{2} f(t)(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

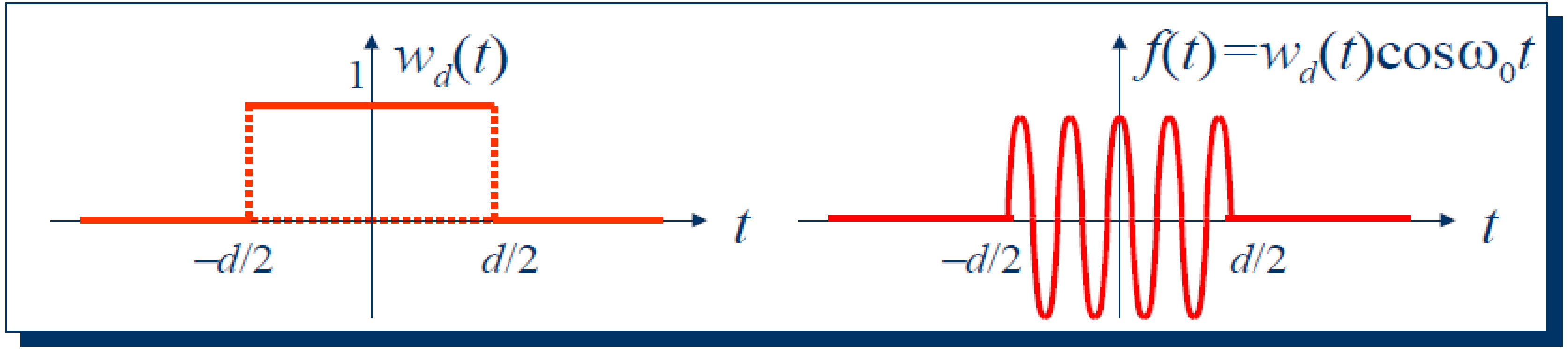
$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}\int_{-\infty}^{\infty} f(t)\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(t)\, e^{-j(\omega-\omega_0)t} dt + \frac{1}{2}\int_{-\infty}^{\infty} f(t)\, e^{-j(\omega+\omega_0)t} dt$$

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}\mathcal{F}[f(t)e^{j\omega_0 t}] + \frac{1}{2}\mathcal{F}[f(t)e^{-j\omega_0 t}]$$

$$= \frac{1}{2}F[j(\omega-\omega_0)] + \frac{1}{2}F[j(\omega+\omega_0)]$$

Example:



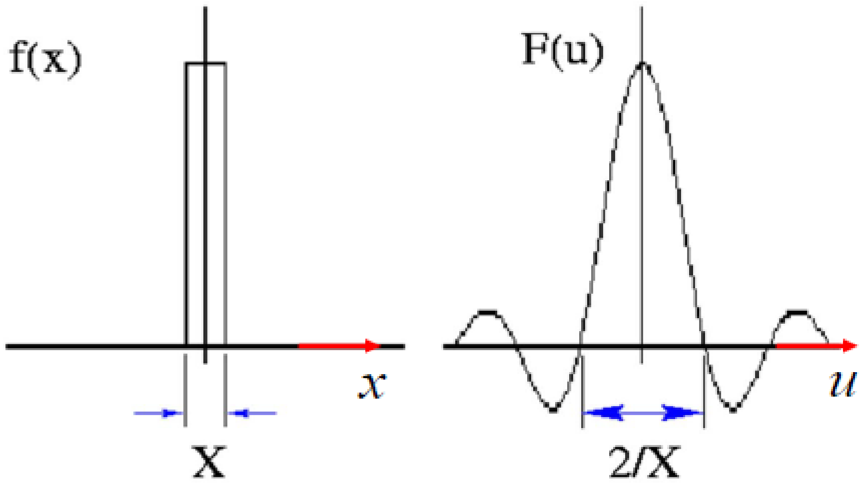
$$W_d(j\omega) = \mathcal{F}[w_d(t)] = \int_{-d/2}^{d/2} e^{-j\omega t}\,dt = \frac{2}{\omega}\sin\left(\frac{\omega d}{2}\right)$$

$$F(j\omega) = \mathcal{F}[w_d(t)\cos\omega_0 t] = \frac{\sin\frac{d}{2}(\omega-\omega_0)}{\omega-\omega_0} + \frac{\sin\frac{d}{2}(\omega+\omega_0)}{\omega+\omega_0}$$

Example

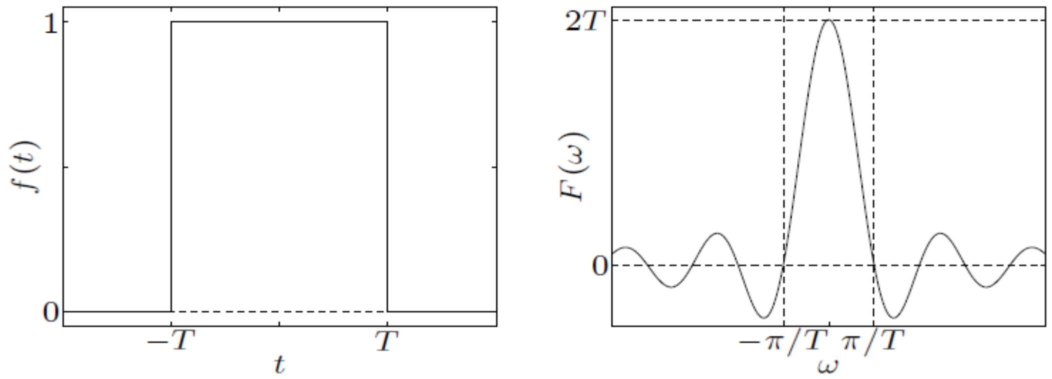
$$f(x) = \begin{cases} 1, \ |x| < \frac{X}{2}, \\ 0, \ |x| \geq \frac{X}{2}. \end{cases}$$

$$\begin{aligned} F(u) &= \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \\ &= \int_{-X/2}^{X/2} e^{-j2\pi ux} dx \\ &= \frac{1}{-j2\pi u}[e^{-j2\pi uX/2} - e^{j2\pi uX/2}] \\ &= X\frac{\sin(\pi X u)}{(\pi X u)} = X\text{sinc}(\pi\text{Xu}). \end{aligned}$$



rectangular pulse: $f(t) = \begin{cases} 1 & -T \le t \le T \\ 0 & |t| > T \end{cases}$

$$F(\omega) = \int_{-T}^{T} e^{-j\omega t}\, dt = \frac{-1}{j\omega}\left(e^{-j\omega T} - e^{j\omega T}\right) = \frac{2\sin\omega T}{\omega}$$



unit impulse: $f(t) = \delta(t)$

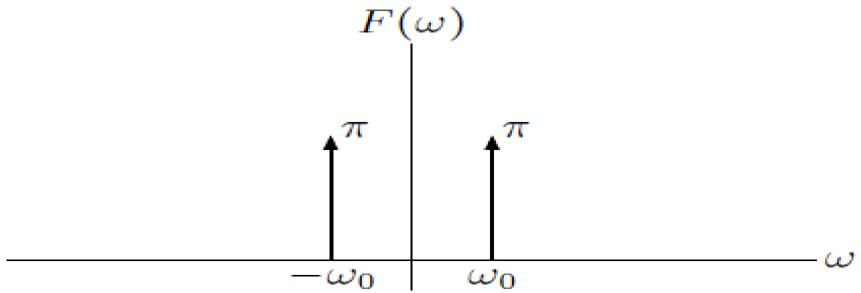
$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = 1$$

Fourier transform of periodic signals

similarly, by allowing impulses in $\mathcal{F}(f)$, we can define the Fourier transform of a periodic signal

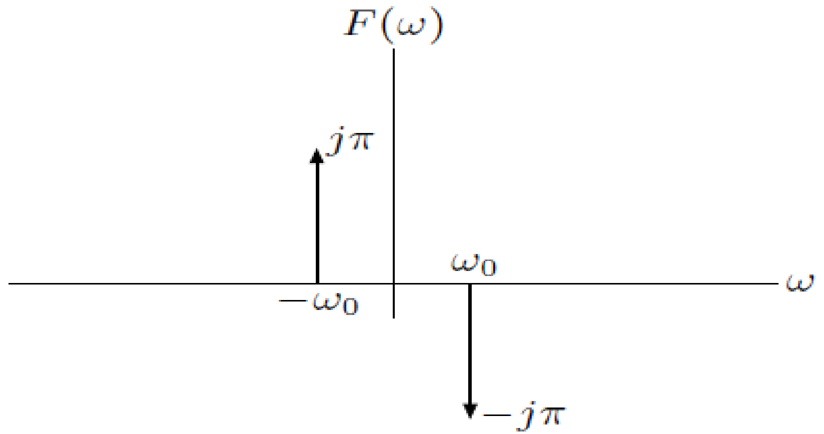
sinusoidal signals: Fourier transform of $f(t) = \cos\omega_0 t$

$$\begin{aligned} F(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j(\omega+\omega_0)t} dt \\ &= \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0) \end{aligned}$$



Fourier transform of $f(t) = \sin \omega_0 t$

$$
\begin{aligned}
F(\omega) &= \frac{1}{2j}\int_{-\infty}^{\infty}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) e^{-j\omega t}\, dt \\
&= \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t} dt + -\frac{1}{2j}\int_{-\infty}^{\infty} e^{-j(\omega_0+\omega)t} dt \\
&= -j\pi\delta(\omega-\omega_0) + j\pi\delta(\omega+\omega_0)
\end{aligned}
$$



Examples

sign function: $f(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$

write f as $f(t) = -1 + 2g(t)$, where g is a unit step at $t = 0$, and apply linearity

$$F(\omega) = -2\pi\delta(\omega) + 2\pi\delta(\omega) + \frac{2}{j\omega} = \frac{2}{j\omega}$$

sinusoidal signal: $f(t) = \cos(\omega_0 t + \phi)$

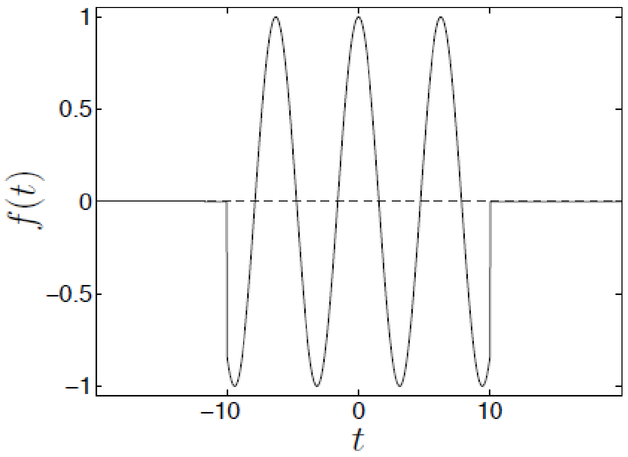
write f as

$$f(t) = \cos(\omega_0(t + \phi/\omega_0))$$

and apply time shift property:

$$F(\omega) = \pi e^{j\omega\phi/\omega_0}\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

pulsed cosine: $f(t) = \begin{cases} 0 & |t| > 10 \\ \cos t & -10 \leq t \leq 10 \end{cases}$



write f as a product $f(t) = g(t)\cos t$ where g is a rectangular pulse of width 20 (see page 12-7)

$$\mathcal{F}(\cos t) = \pi\delta(\omega - 1) + \pi\delta(\omega + 1), \quad \mathcal{F}(g(t)) = \frac{2\sin 10\omega}{\omega}$$

The Fourier series for the function $f(x) = \sin^2 x$ is

$$f(x) = \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= 0.5 - 0.5 \ \cos 2x$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 x + b_n \sin n\omega_0 x$$

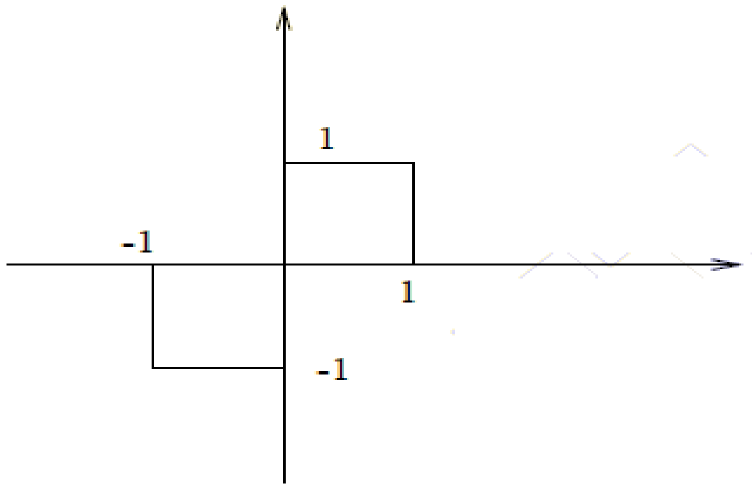
$f(x) = \sin^2 x$ is an even function so $b_n = 0$

$$A_0 = 0.5$$

$$a_n = \begin{cases} -0.5, & n = 1 \\ 0 & , \text{otherwise} \end{cases}$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{T} = 2$$

$$x(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$



$$\begin{aligned} X(\omega) &= -\int_{-1}^{0} e^{-j\omega t} dt + \int_{0}^{1} e^{-j\omega t} dt \\ &= -\int_{0}^{1} e^{j\omega t} dt + \int_{0}^{1} e^{-j\omega t} dt \\ &= -2j \int_{0}^{1} \sin(\omega t)\, dt \\ &= 2j \frac{1}{\omega} \cos(\omega t) \Big|_{0}^{1} \quad = \quad 2j \frac{1}{\omega} (\cos(\omega) - 1) \end{aligned}$$

Find the Fourier transform of $x(t) = f(t - 2) + f(t + 2)$.

Using linearity property,
$ax(t) + by(t) \leftrightarrow aX(\omega) + bY(\omega)$
And Time shifting property,
$f(t-t_0) \leftrightarrow e^{-j\omega_0 t} F(j\omega)$,
We have $F[x(t)] = F[f(t)]\, e^{-j2\omega} + F[f(t)]\, e^{j2\omega}$
$= F(j\omega)e^{-j2\omega} + F(j\omega)e^{j2\omega} = 2F(j\omega)\cos 2\omega$.

Find the Fourier transform of $e^{j\omega_0 t}$.

Explanation:
We know that $F[1] = 2\pi\delta(\omega)$
By using the frequency shifting property,
$e^{j\omega_0 t}\, x(t) \leftrightarrow X(\omega - \omega_0)$
We have $F[e^{j\omega_0 t}] = F[e^{j\omega_0 t}\,(1)] = 2\pi\delta(\omega - \omega_0)$.

Find the Fourier transform of $f(t)=te^{-at}\, u(t)$.

Using frequency differentiation property, $tx(t) \leftrightarrow j\frac{d}{d\omega}X(\omega)$

$$F[te^{-at}u(t)] = j\frac{d}{d\omega}F[te^{-at}\,u(t)] = j\frac{d}{d\omega}\frac{1}{a+j\omega} = j\frac{-1(j)}{(a+j\omega)^2} = \frac{1}{(a+j\omega)^2}$$

$$te^{-at}\,u(t) \leftrightarrow \frac{1}{(a+j\omega)^2}.$$

The Fourier transform of a Gaussian pulse is also a Gaussian pulse.

Explanation: Gaussian pulse, $x(t) = e^{-\pi t^2}$
Its Fourier transform is $X(f) = e^{-\pi f^2}$
Hence, the Fourier transform of a Gaussian pulse is also a Gaussian pulse.

Show that f(x) = 1, $0 < x < \infty$ cannot be represented by a Fourier integral.

$\int_0^\infty |f(x)|\,dx = \int_0^\infty 1\,dx = [x]_0^\infty = \infty$ and this value tends to ∞ as $x \to \infty$.

i.e., $\int_0^\infty 1 f(x)\,dx$ is not convergent. Hence $f(x) = 1$ cannot be represented by a Fourier integral.

2D Fourier transform

Definition

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy,$$
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}\,du\,dv$$

where u and v are spatial frequencies.

Also will write FT pairs as $f(x,y) \Leftrightarrow F(u,v)$.

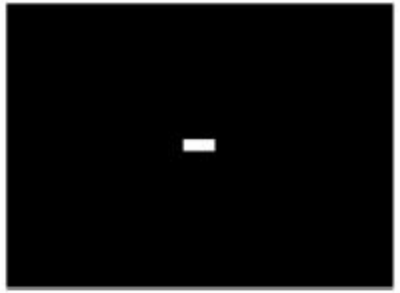
- $F(u,v)$ is complex in general,

$$F(u,v) = F_{\mathrm{R}}(u,v) + jF_{\mathrm{I}}(u,v)$$

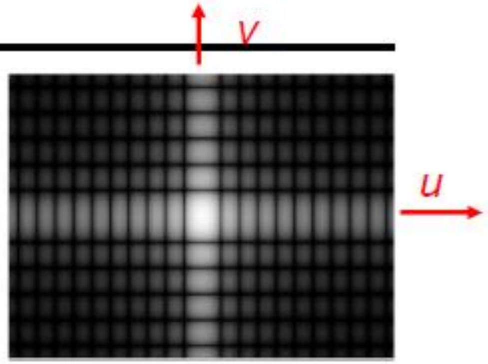
- $|F(u,v)|$ is the magnitude spectrum
- $\arctan(F_{\mathrm{I}}(u,v)/F_{\mathrm{R}}(u,v))$ is the phase angle spectrum.
- Conjugacy: $f^*(x,y) \Leftrightarrow F(-u,-v)$
- Symmetry: $f(x,y)$ is even if $f(x,y) = f(-x,-y)$

FT pair example 1

rectangle centred at origin with sides of length X and Y



$f(x,y)$



$|F(u,v)|$

$$F(u,v) = \int\int f(x,y)e^{-j2\pi(ux+vy)}dxdy,$$

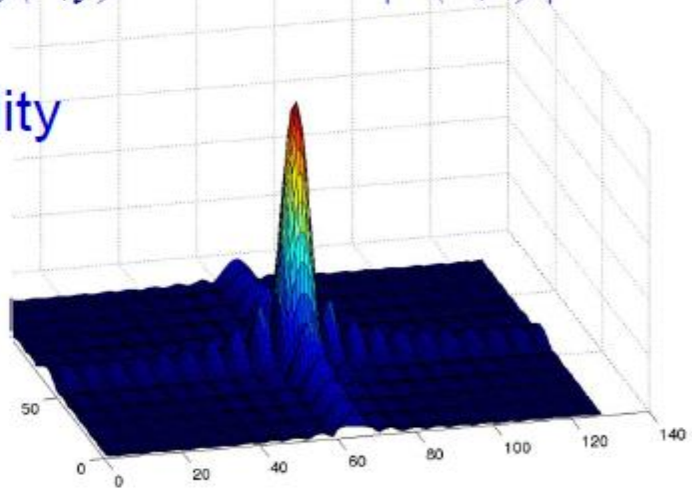
$$= \int_{-X/2}^{X/2} e^{-j2\pi ux}dx \int_{-Y/2}^{Y/2} e^{-j2\pi vy}dy,$$ separability

$$= \left[\frac{e^{-j2\pi ux}}{-j2\pi u}\right]_{-X/2}^{X/2} \left[\frac{e^{-j2\pi vy}}{-j2\pi v}\right]_{-Y/2}^{Y/2},$$

$$= \frac{1}{-j2\pi u}\left[e^{-juX} - e^{juX}\right] \frac{1}{-j2\pi v}\left[e^{-jvY} - e^{jvY}\right],$$

$$= XY\left[\frac{sin(\pi Xu)}{\pi Xu}\right]\left[\frac{sin(2\pi Yv)}{\pi Yv}\right]$$

$$= XY\mathrm{sinc}(\pi \mathrm{Xu})\mathrm{sinc}(\pi \mathrm{Yv}).$$



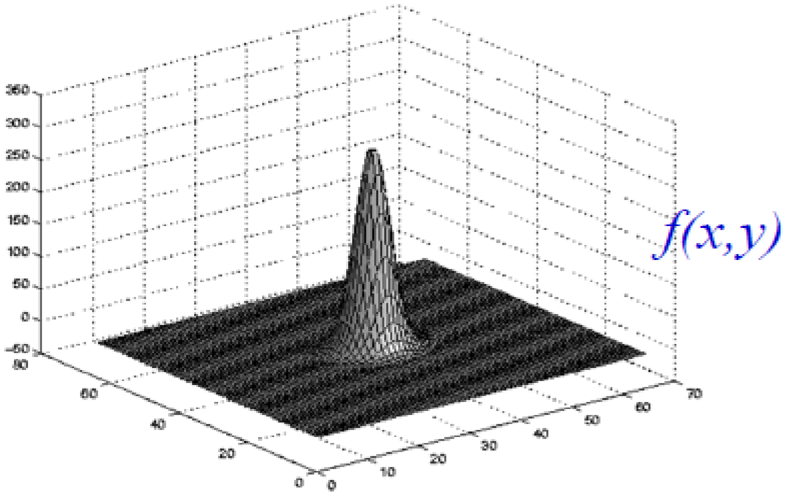
$|F(u,v)|$

FT pair example 2

Gaussian centred on origin

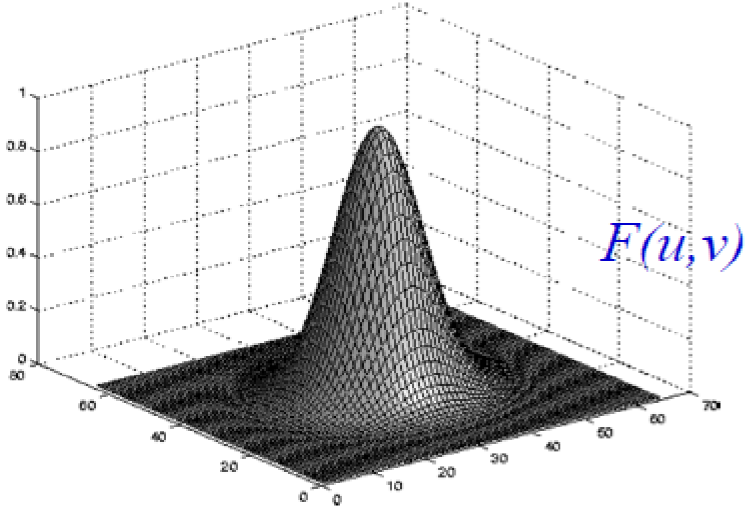
$$f(r) = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$

where $r^2 = x^2 + y^2$.



$$F(u,v) = F(\rho) = e^{-2\pi^2\rho^2\sigma^2}$$

where $\rho^2 = u^2 + v^2$.

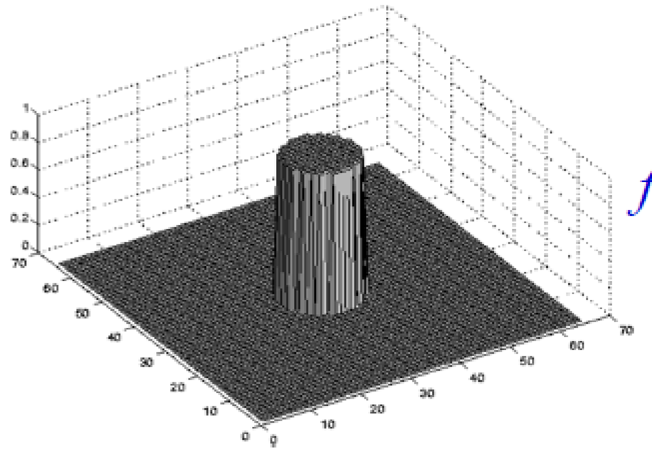


- FT of a Gaussian is a Gaussian
- Note inverse scale relation

FT example

Circular disk unit height and radius *a* centred on origin

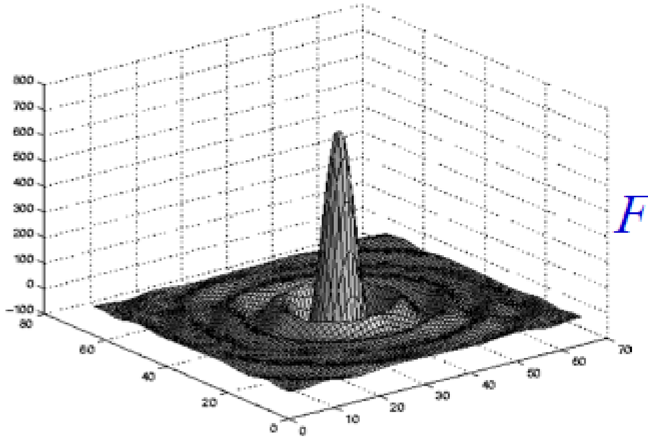
$$f(x,y) = \begin{cases} 1, & |r| < a, \\ 0, & |r| \geq a. \end{cases}$$



f(x,y)

$$F(u,v) = F(\rho) = aJ_1(\pi a \rho)/\rho$$

where $J_1(x)$ is a Bessel function.



F(u,v)

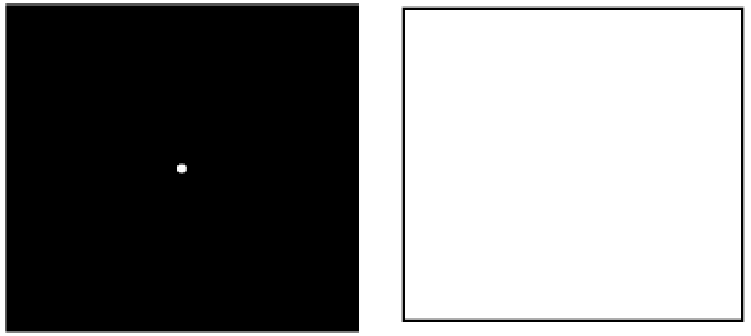
- rotational symmetry
- a ‘2D’ version of a sinc

FT example

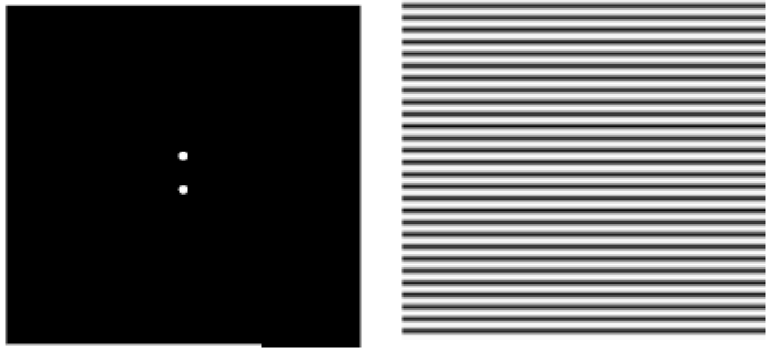
$$f(x,y) = \delta(x,y) = \delta(x)\delta(y)$$

$$F(u,v) = \int\int \delta(x,y) e^{-j2\pi(ux+vy)}\, dx dy$$

$$= 1$$



f(x,y) *F(u,v)*



$$f(x,y) = \frac{1}{2}\left(\delta(x, y-a) + \delta(x, y+a)\right)$$

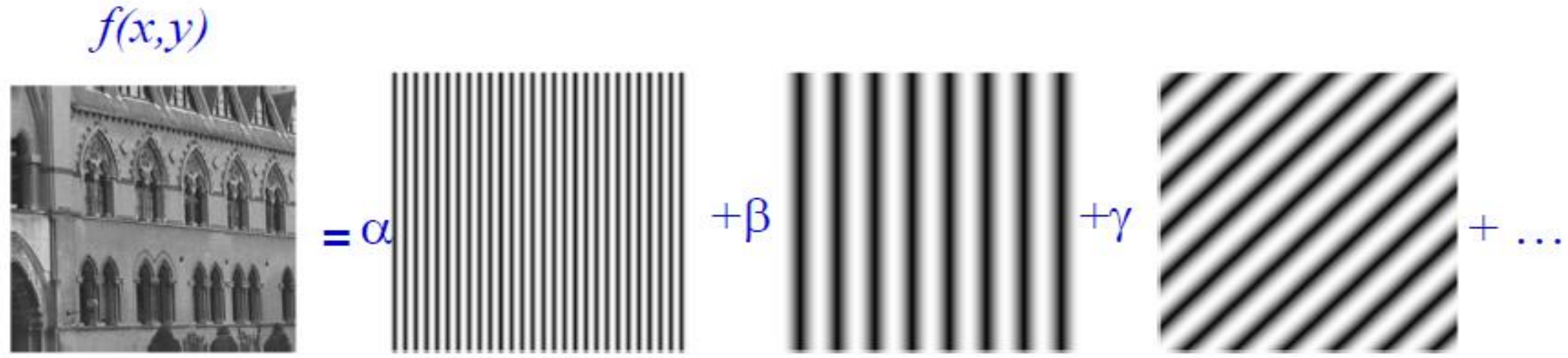
$$F(u,v) = \frac{1}{2}\int\int \left(\delta(x,y-a) + \delta(x,y+a)\right) e^{-j2\pi(ux+vy)}\, dx dy$$

$$= \frac{1}{2}\left(e^{-j2\pi av} + e^{j2\pi av}\right) = \cos 2\pi av$$

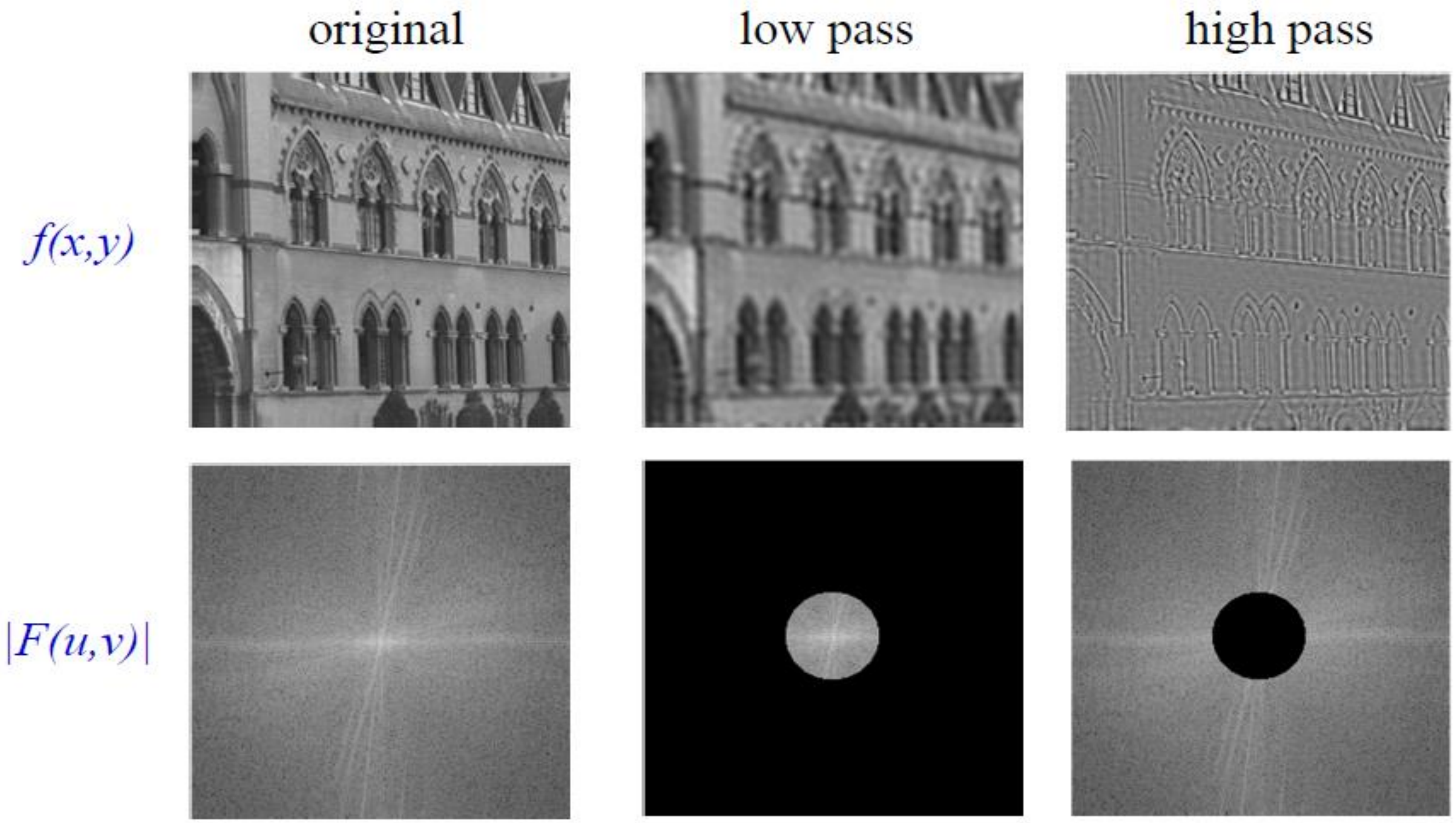
The spatial function $f(x, y)$

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} \, du \, dv$$

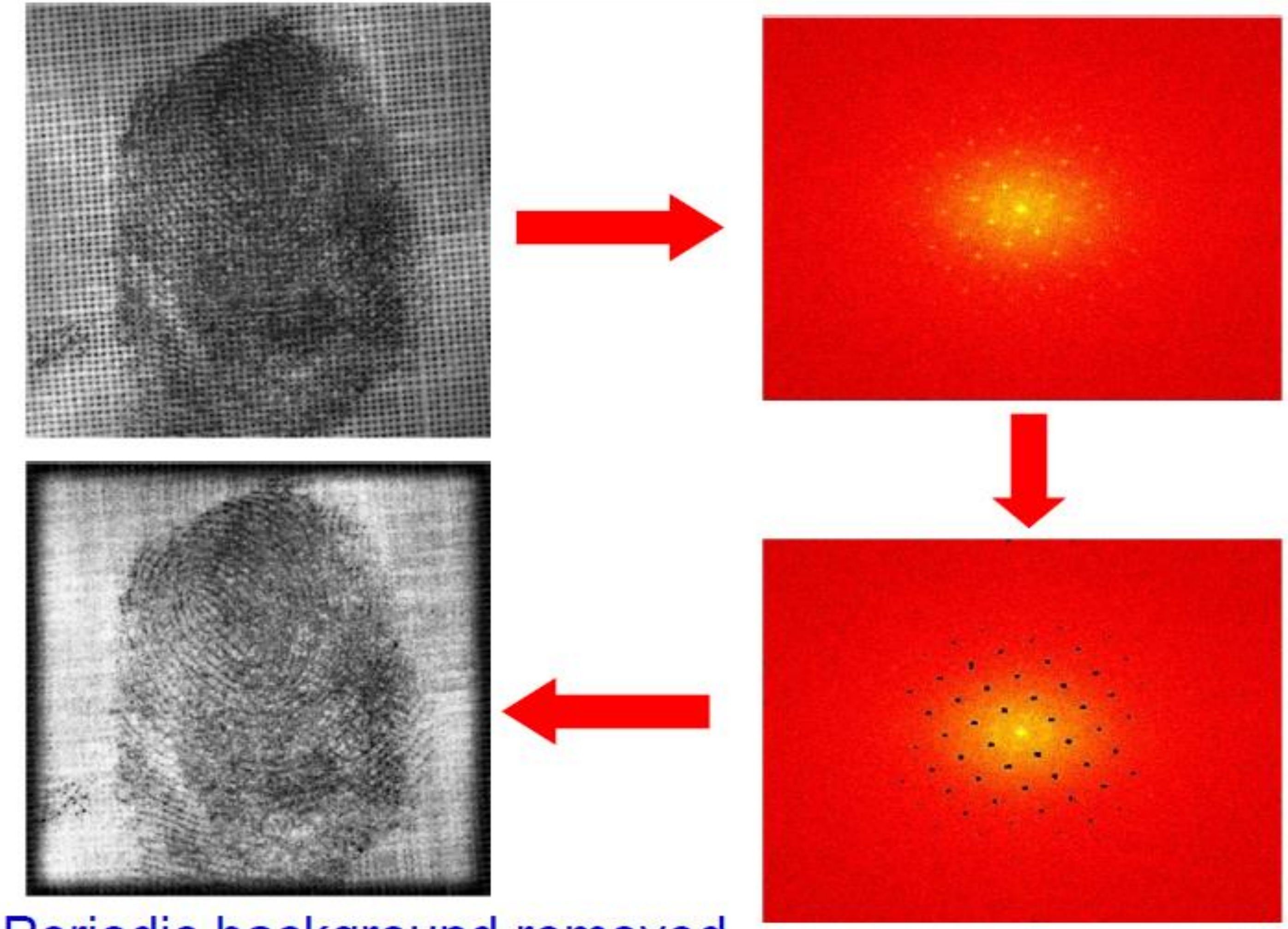
is decomposed into a weighted sum of 2D orthogonal basis functions in a similar manner to decomposing a vector onto a basis using scalar products.



Example: action of filters on a real image



Example – Forensic application



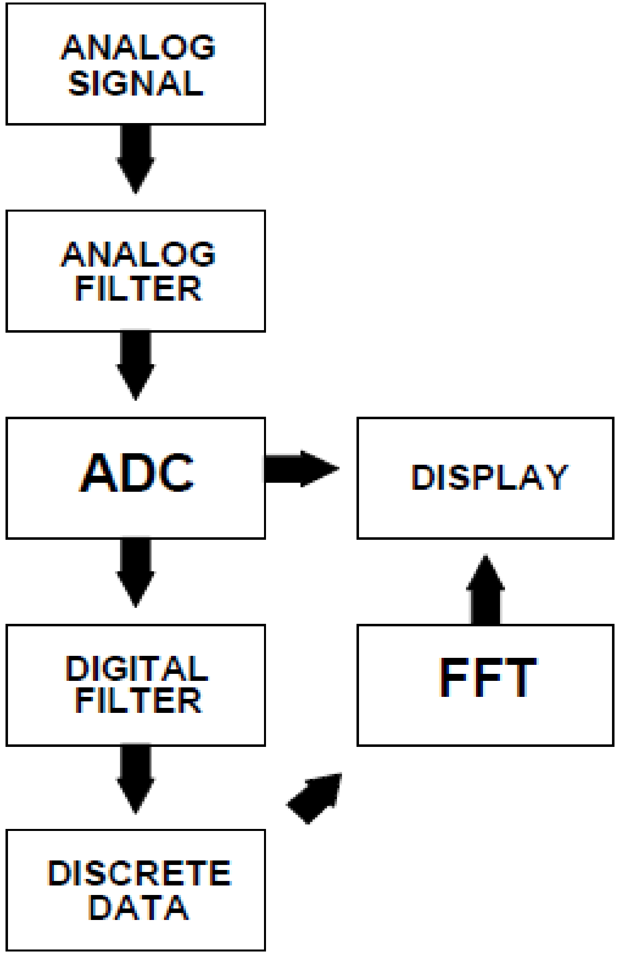
$|F(u,v)|$

remove peaks

Periodic background removed

FFT
&
Signal Analysis

The Anatomy of the FFT Analyzer



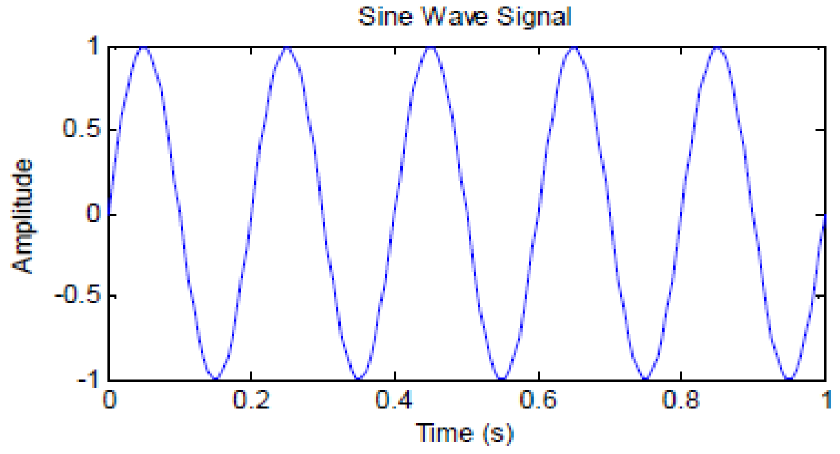
The FFT Analyzer can be broken dowr into several pieces which involve the digitization, filtering, transformation and processing of a signal.

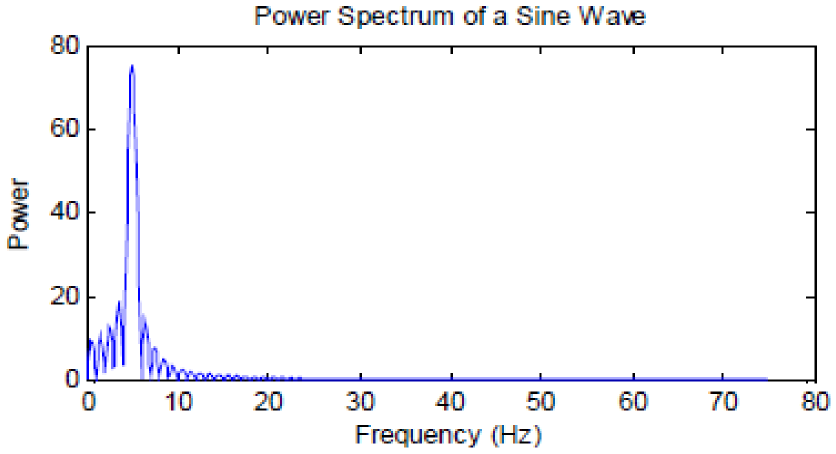
Several items are important here:

- Digitization and Sampling
- Quantization of Signal
- Aliasing Effects
- Leakage Distortion
- Windows Weighting Functions
- The Fourier Transform
- Measurement Formulation

Quantity	Description
x	Sampled data
n = length(x)	Number of samples
fs	Sample frequency (samples per unit time or space)
dt = 1/fs	Time or space increment per sample
t = (0:n-1)/fs	Time or space range for data
y = fft(x)	Discrete Fourier transform of data (DFT)
abs(y)	Amplitude of the DFT
(abs(y).^2)/n	Power of the DFT
fs/n	Frequency increment
f = (0:n-1)*(fs/n)	Frequency range
fs/2	Nyquist frequency (midpoint of frequency range)

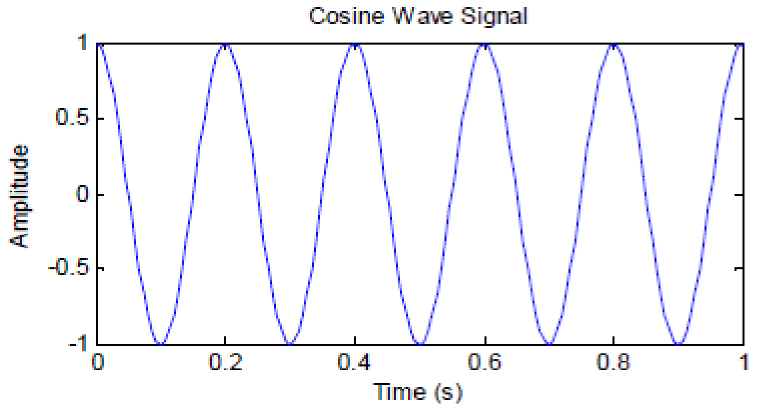
Sine Wave

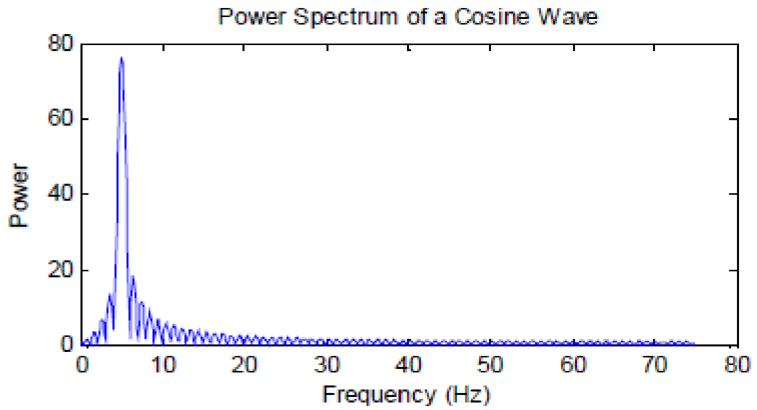





```
Fs = 150; % Sampling frequency
t = 0:1/Fs:1; % Time vector of 1 second
f = 5; % Create a sine wave of f Hz.
x = sin(2*pi*t*f);
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that length(X)
is equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% Frequency vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Sine Wave Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of a Sine Wave');
xlabel('Frequency (Hz)');
ylabel('Power');
```

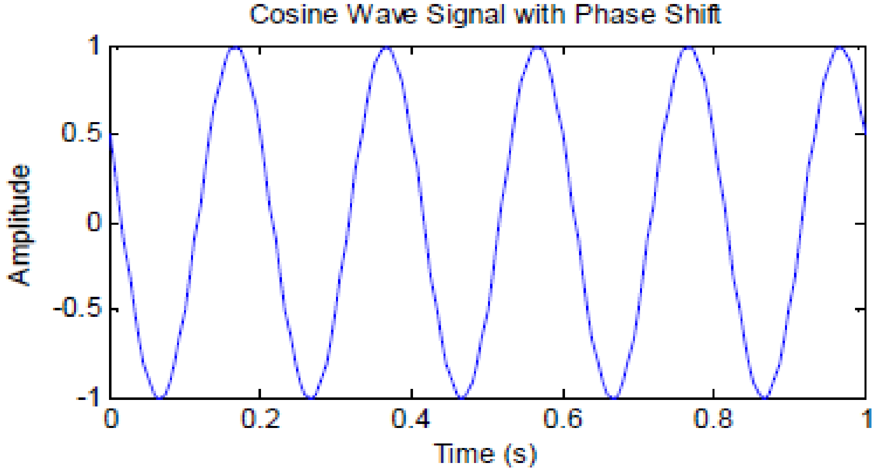
Cosine Wave

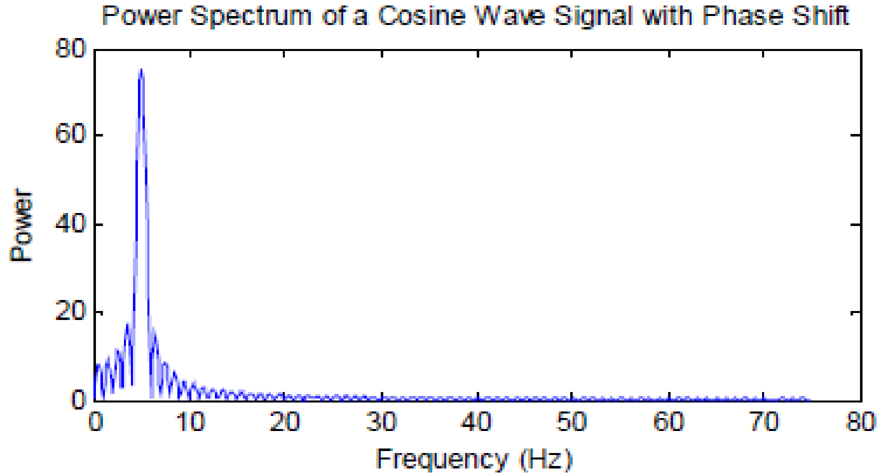





```
Fs = 150; % Sampling frequency
t = 0:1/Fs:1; % Time vector of 1 second
f = 5; % Create a sine wave of f Hz.
x = cos(2*pi*t*f);
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that length(X) is
equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% Frequency vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Sine Wave Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of a Sine Wave');
xlabel('Frequency (Hz)');
ylabel('Power');
```

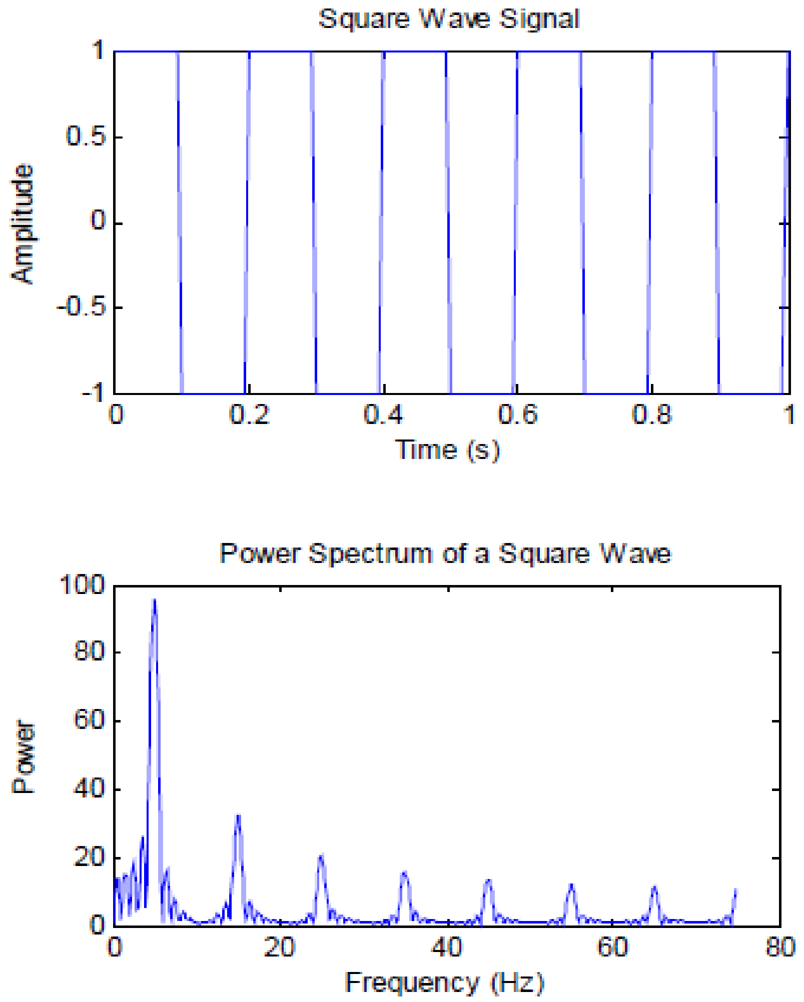
Cosine Wave with Phase Shift





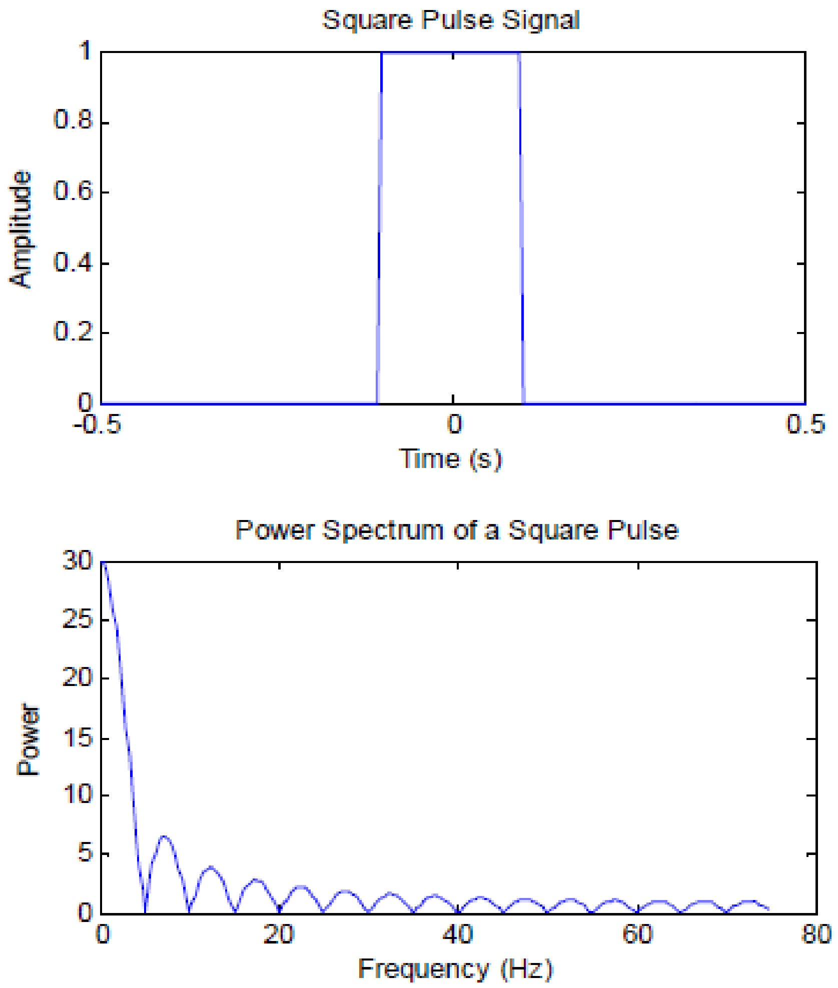

```
Fs = 150; % Sampling frequency
t = 0:1/Fs:1; % Time vector of 1 second
f = 5; % Create a sine wave of f Hz.
pha = 1/3*pi; % phase shift
x = cos(2*pi*t*f + pha);
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that length(X) is
equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% Frequency vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Sine Wave Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of a Sine Wave');
xlabel('Frequency (Hz)');
ylabel('Power');
```

Square Wave



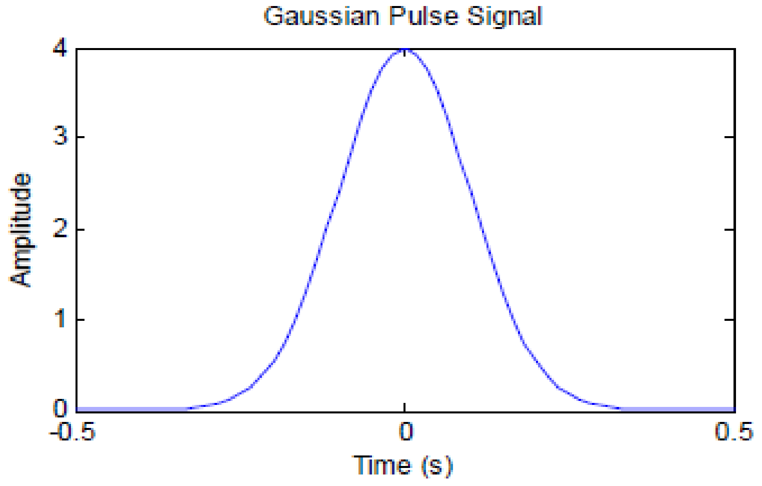

```
Fs = 150; % Sampling frequency
t = 0:1/Fs:1; % Time vector of 1 second
f = 5; % Create a sine wave of f Hz.
x = square(2*pi*t*f);
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that length(X) is
equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% Frequency vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Square Wave Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of a Square Wave');
xlabel('Frequency (Hz)');
ylabel('Power');
```

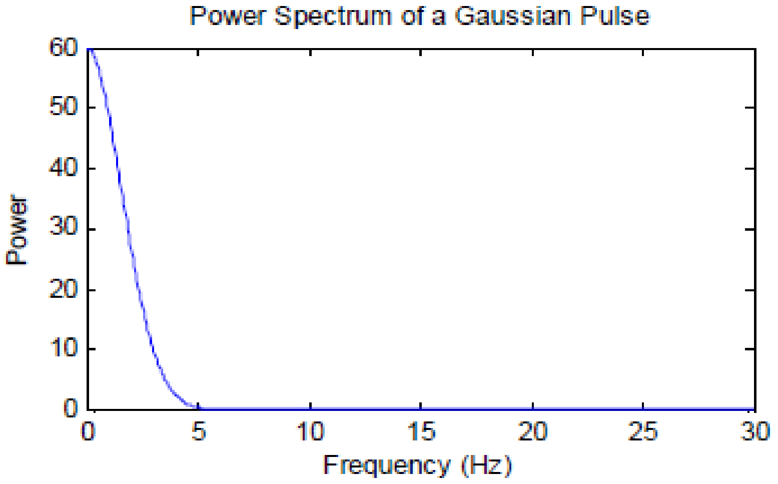
Square Pulse




```
Fs = 150; % Sampling frequency
t = -0.5:1/Fs:0.5; % Time vector of 1 second
w = .2; % width of rectangle
x = rectpuls(t, w); % Generate Square Pulse
nfft = 512; % Length of FFT
% Take fft, padding with zeros so that length(X) is
equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% Frequency vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Square Pulse Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of a Square Pulse');
xlabel('Frequency (Hz)');
ylabel('Power');
```

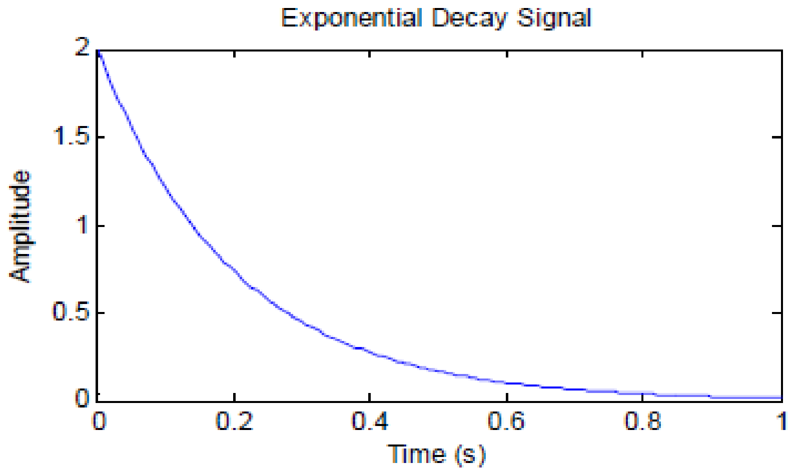
Gaussian Pulse

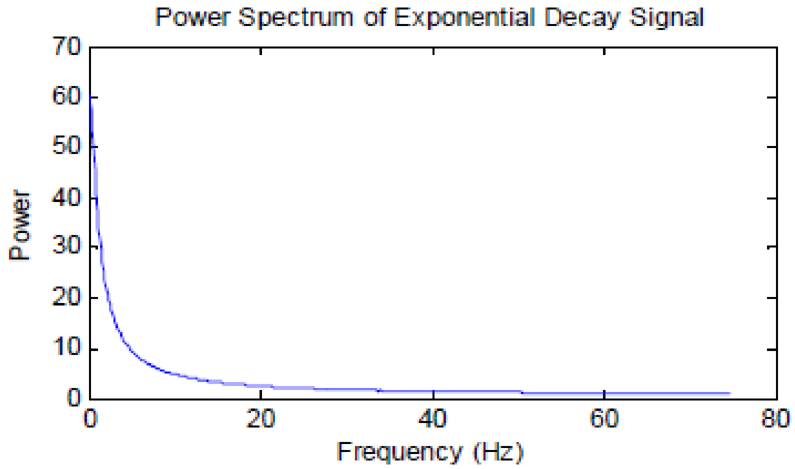





```
Fs = 60; % Sampling frequency
t = -.5:1/Fs:.5;
x = 1/(sqrt(2*pi*0.01))*(exp(-t.^2/(2*0.01)));
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that
length(X) is equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% This is an evenly spaced frequency vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Gaussian Pulse Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of a Gaussian Pulse');
xlabel('Frequency (Hz)');
ylabel('Power');
```

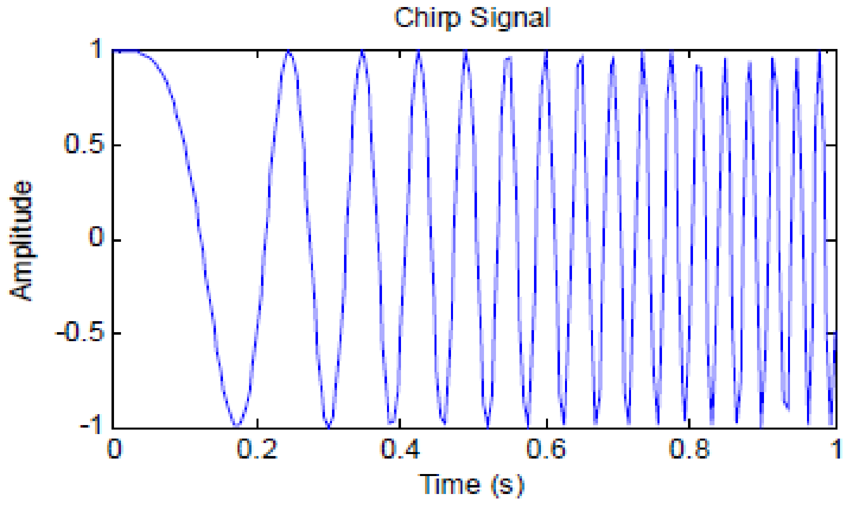
Exponential Decay

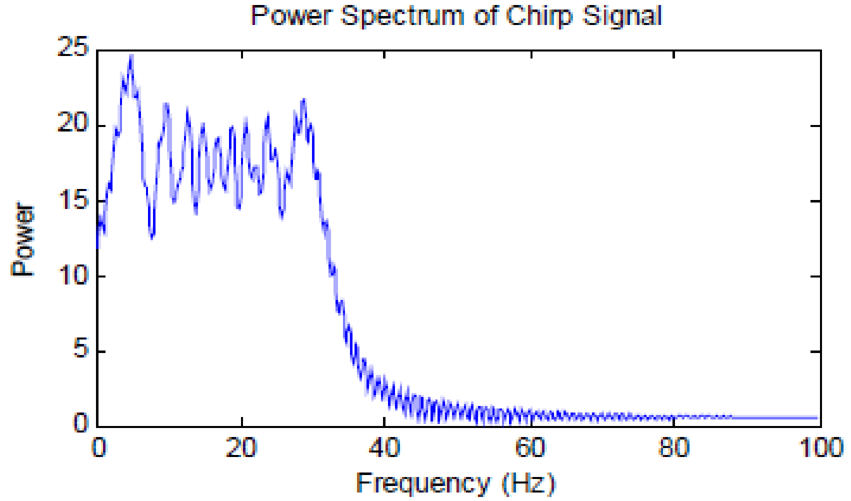





```
Fs = 150; % Sampling frequency
t = 0:1/Fs:1; % Time vector of 1 second
x = 2*exp(-5*t);
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that
length(X) is equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second
half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% This is an evenly spaced frequency
vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Exponential Decay Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of Exponential
Decay Signal');
xlabel('Frequency (Hz)');
ylabel('Power');
```

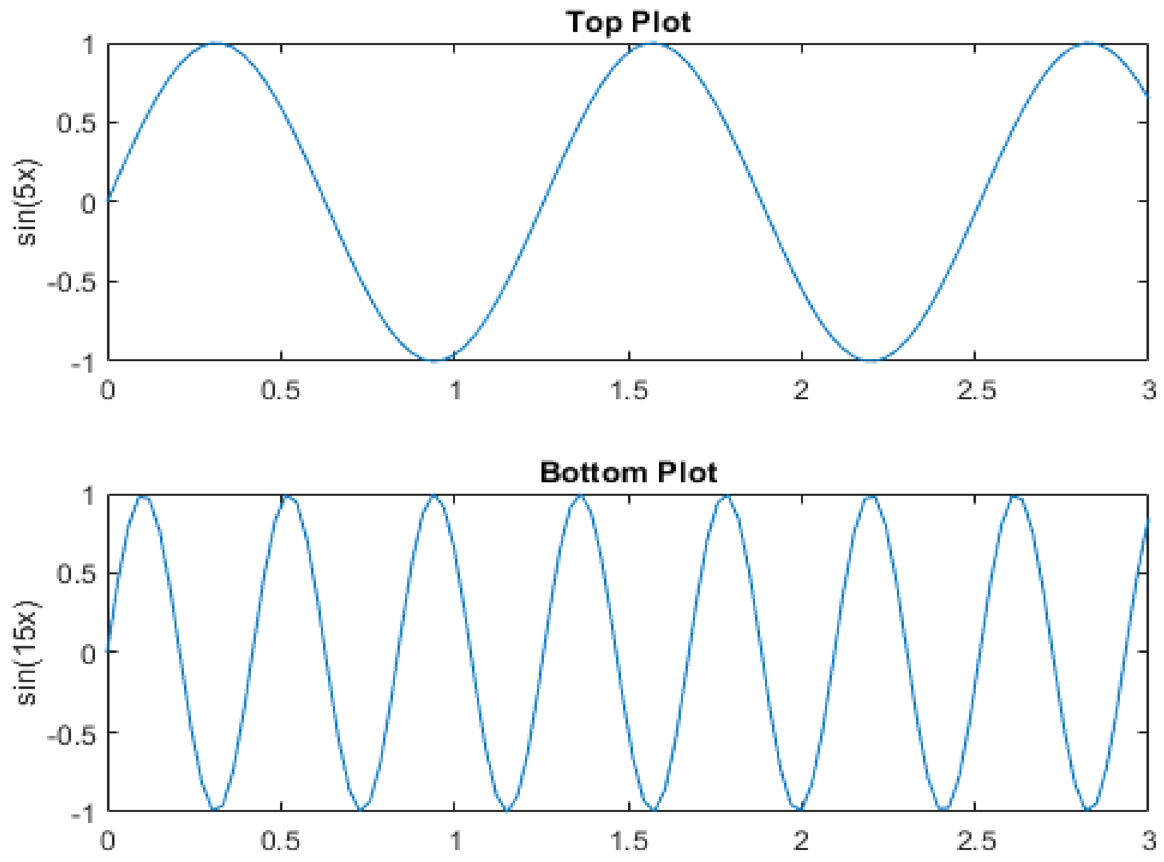
Chirp Signal



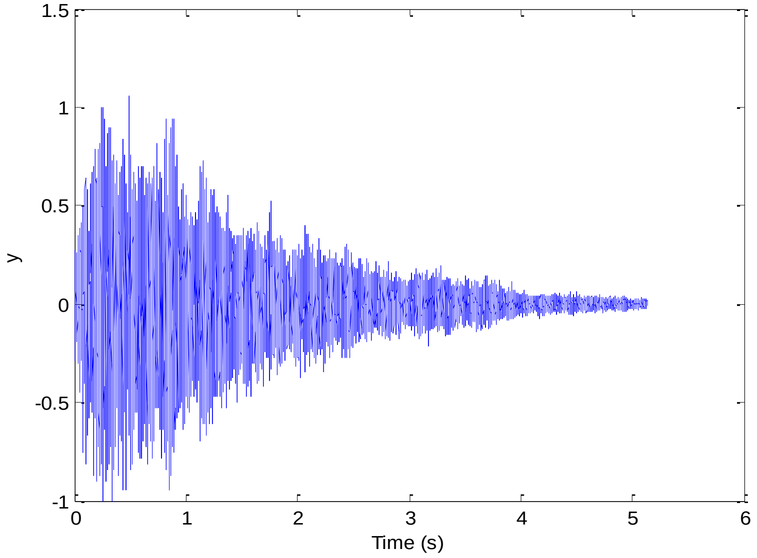


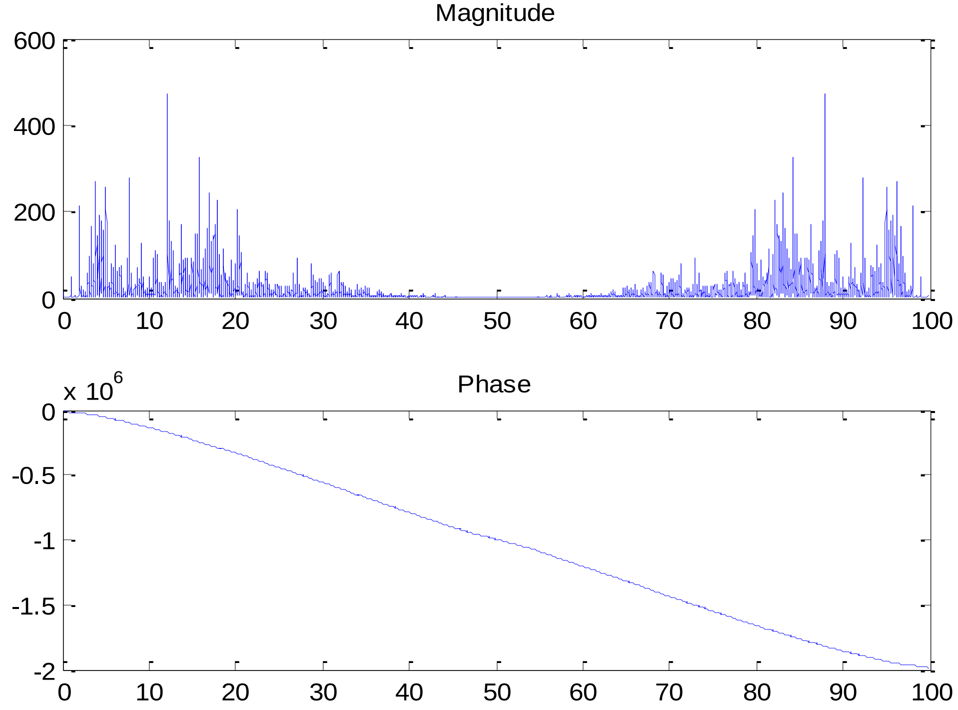

```
Fs = 200; % Sampling frequency
t = 0:1/Fs:1; % Time vector of 1 second
x = chirp(t,0,1,Fs/6);
nfft = 1024; % Length of FFT
% Take fft, padding with zeros so that
length(X) is equal to nfft
X = fft(x,nfft);
% FFT is symmetric, throw away second half
X = X(1:nfft/2);
% Take the magnitude of fft of x
mx = abs(X);
% This is an evenly spaced frequency
vector
f = (0:nfft/2-1)*Fs/nfft;
% Generate the plot, title and labels.
figure(1);
plot(t,x);
title('Chirp Signal');
xlabel('Time (s)');
ylabel('Amplitude');
figure(2);
plot(f,mx);
title('Power Spectrum of Chirp Signal');
xlabel('Frequency (Hz)');
ylabel('Power');
```

```
% Create data and 2-by-1 tiled chart layout
x = linspace(0,3);
y1 = sin(5*x);
y2 = sin(15*x);
tiledlayout(2,1)

% Top plot
ax1 = nexttile;
plot(ax1,x,y1)
title(ax1,'Top Plot')
ylabel(ax1,'sin(5x)')

% Bottom plot
ax2 = nexttile;
plot(ax2,x,y2)
title(ax2,'Bottom Plot')
ylabel(ax2,'sin(15x)')
```



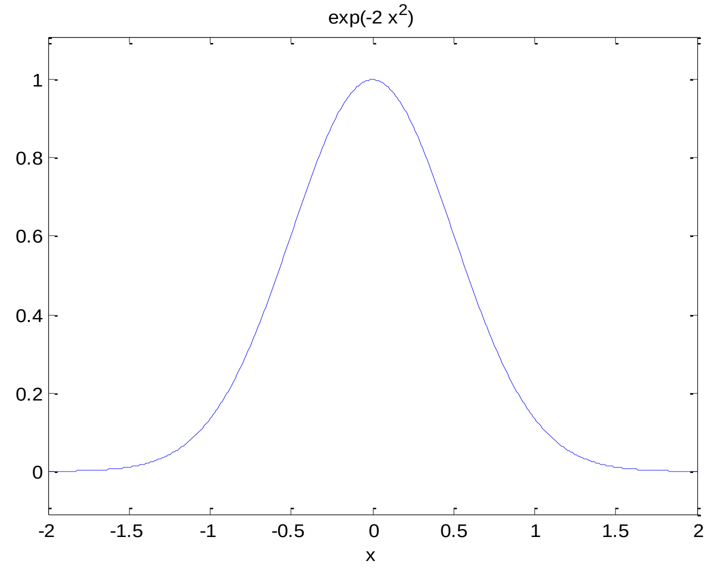
```
clear all
close all
load('gong') %load the variables for the 'gong' audio file, this loads
the sample frequency and the sample values
Fs
t=0:1/Fs:length(y)/Fs-1/Fs; %time index
figure;plot(t,y);xlabel('Time (s)'),ylabel('y')
y1 = fft(y);                           % Compute DFT of x
m = abs(y1);                             % Magnitude
y1(m<1e-6) = 0;
p = unwrap(angle(y1));                   % Phase
f = (0:length(y1)-1)*100/length(y1);
figure,subplot(2,1,1)
plot(f,m)
title('Magnitude')
ax = gca;
ax.XTick = [15 40 60 85];
subplot(2,1,2)
plot(f,p*180/pi)
title('Phase')
ax = gca;
ax.XTick = [15 40 60 85]
```

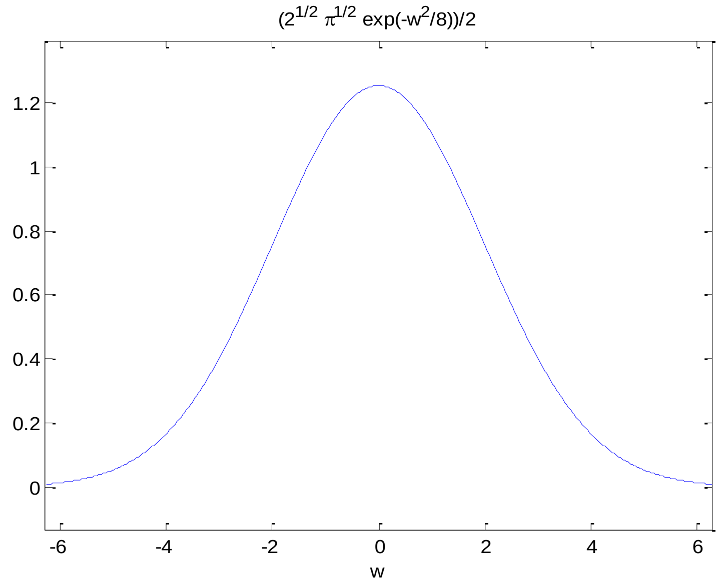




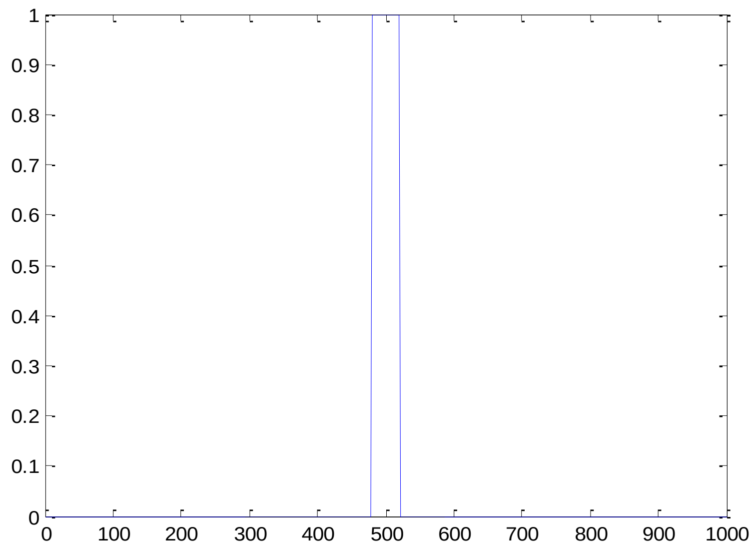
```
clear all
close all
syms x
f = exp(-2*x^2);     %our function
figure, ezplot(f,[-2,2])     % plot of our function
FT = fourier(f)     % Fourier transform

figure, ezplot(FT)
```

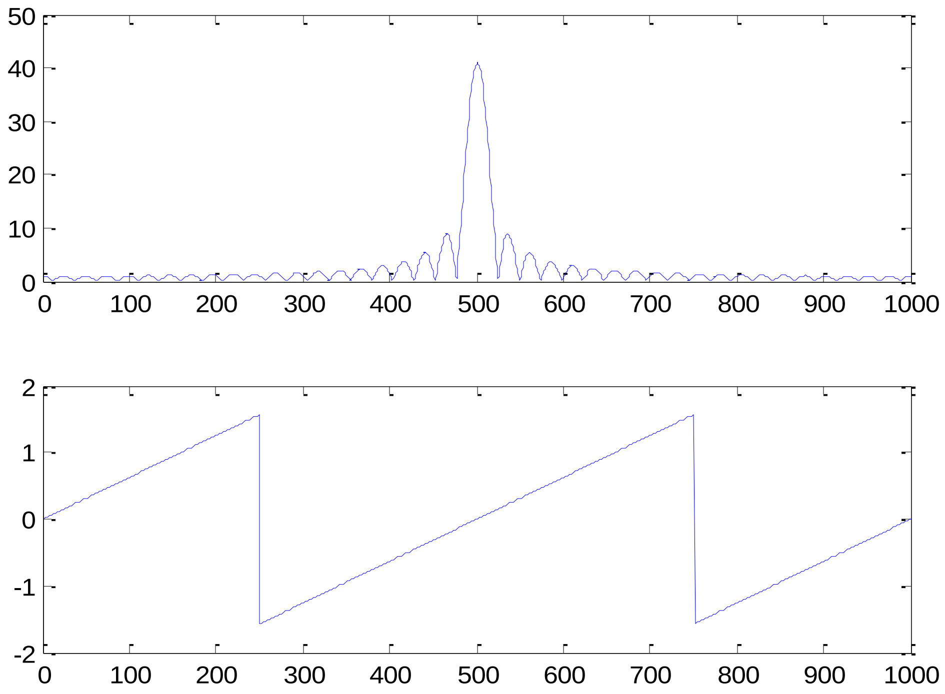
FT = (2^(1/2)*pi^(1/2)*exp(-w^2/8))/2



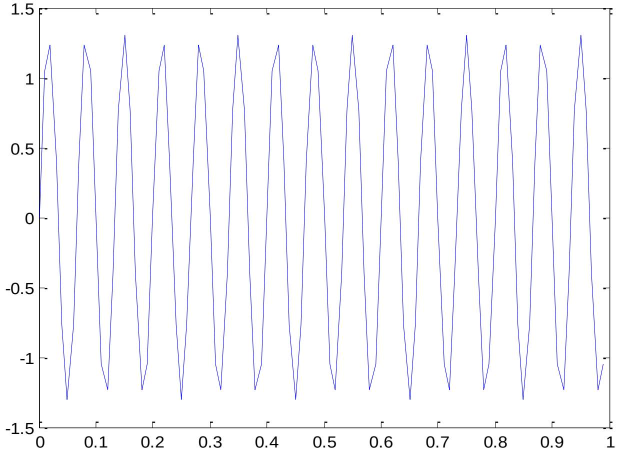


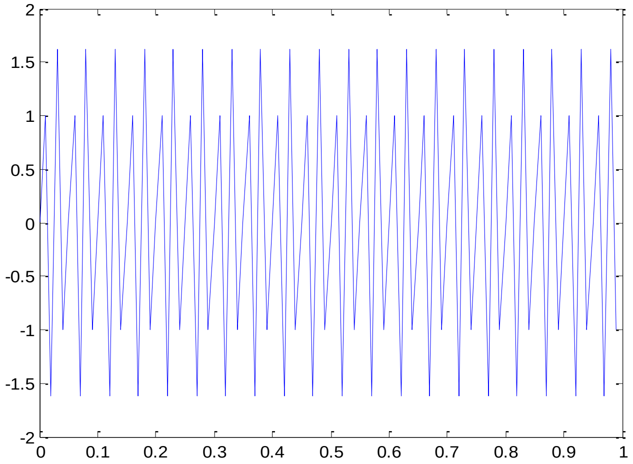
```
clear all
close all


M = 1000;
f = zeros(1, M);
l = 20;
f(M/2-l:M/2+l) = 1;
F = fft(f);
Fc = fftshift(F);
rFc = real(Fc);
iFc = imag(Fc);
subplot(2,1,1),plot(abs(Fc));
subplot(2,1,2),plot(atan(iFc./rFc));
```

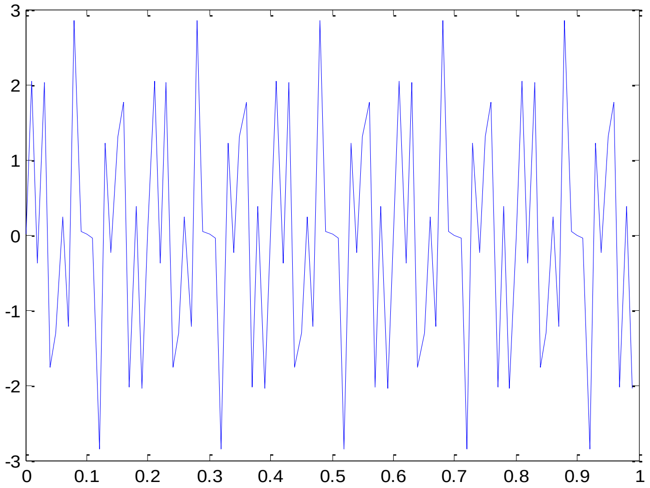


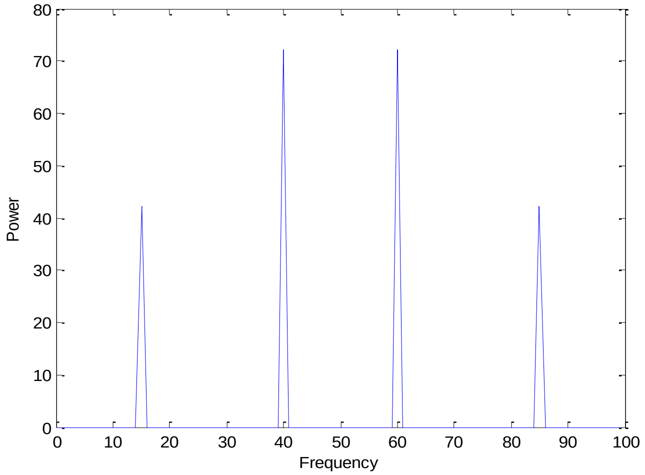


```
clear all
close all

fs = 100;                          % sample frequency (Hz)
t = 0:1/fs:1-1/fs;                 % 1 second span time vector
x1 = (1.3)*sin(2*pi*15*t);         %  15 Hz component,
figure, plot(t,x1)
x2 = (1.7)*sin(2*pi*40*(t-2));     %   40 Hz component
figure, plot(t,x2)
x=x1+x2;
figure, plot(t,x)

y = fft(x);
n = length(x);          % number of samples
f = (0:n-1)*(fs/n);     % frequency range
power = abs(y).^2/n;    % power of the DFT

figure, plot(f,power)
xlabel('Frequency')
ylabel('Power')
```

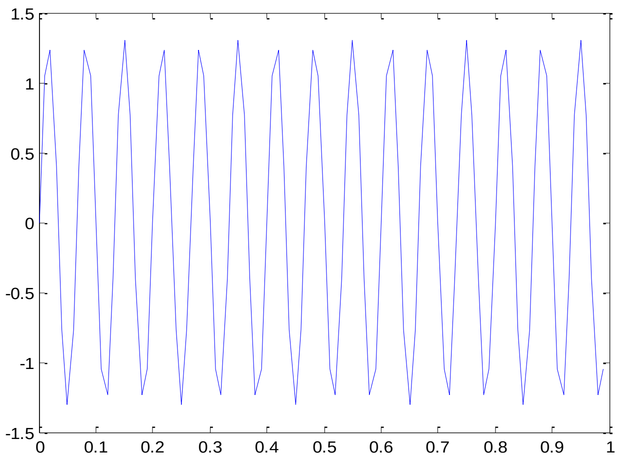


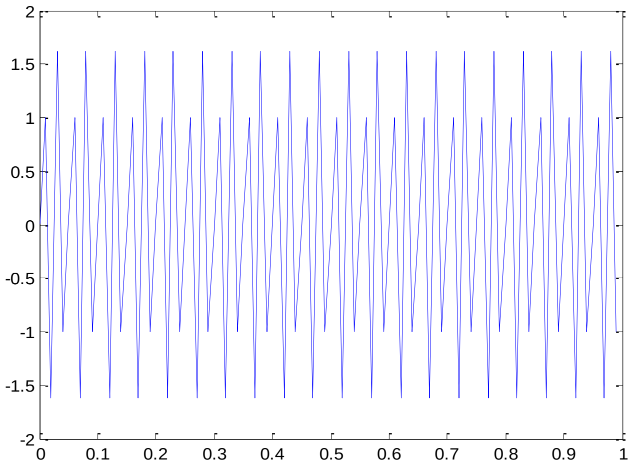


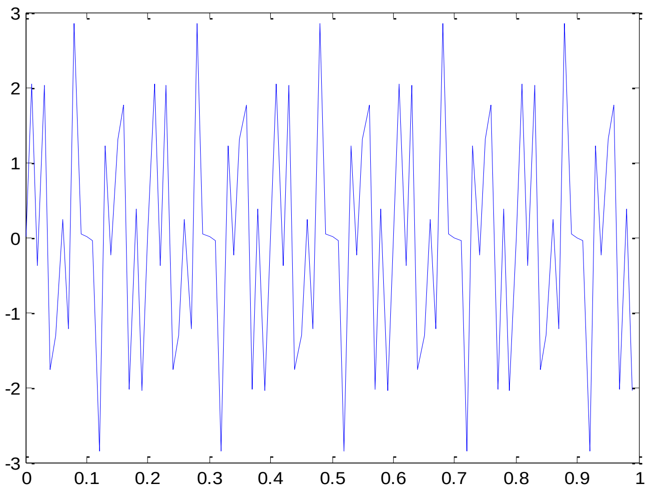


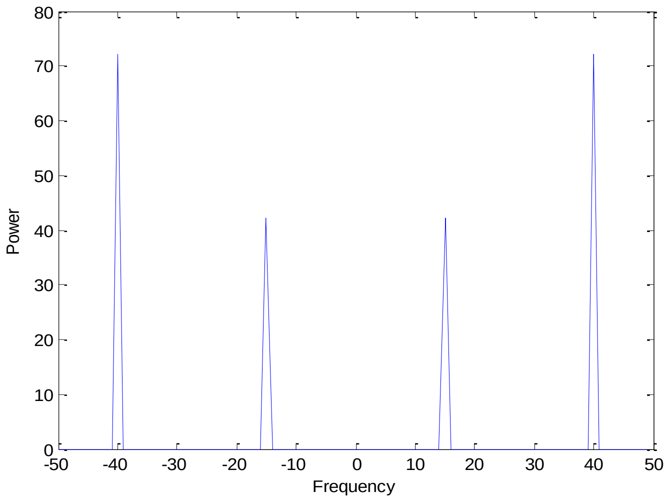


```
clear all
close all

 fs = 100;                              % sample frequency (100Hz)
 t = 0:1/fs:1-1/fs;                     % 1 second span time vector
 x1 = (1.3)*sin(2*pi*15*t);             %  15 Hz component,
 figure, plot(t,x1)
 x2 = (1.7)*sin(2*pi*40*(t-2));         %   40 Hz component
 figure, plot(t,x2)
 x=x1+x2;
 figure, plot(t,x)

 y = fft(x);
 n = length(x);         % number of samples
y0 = fftshift(y);        % shift y values
f0 = (-n/2:n/2-1)*(fs/n); % 0-centered frequency range
power0 = abs(y0).^2/n;    % 0-centered power

plot(f0,power0)
xlabel('Frequency')
ylabel('Power')
```

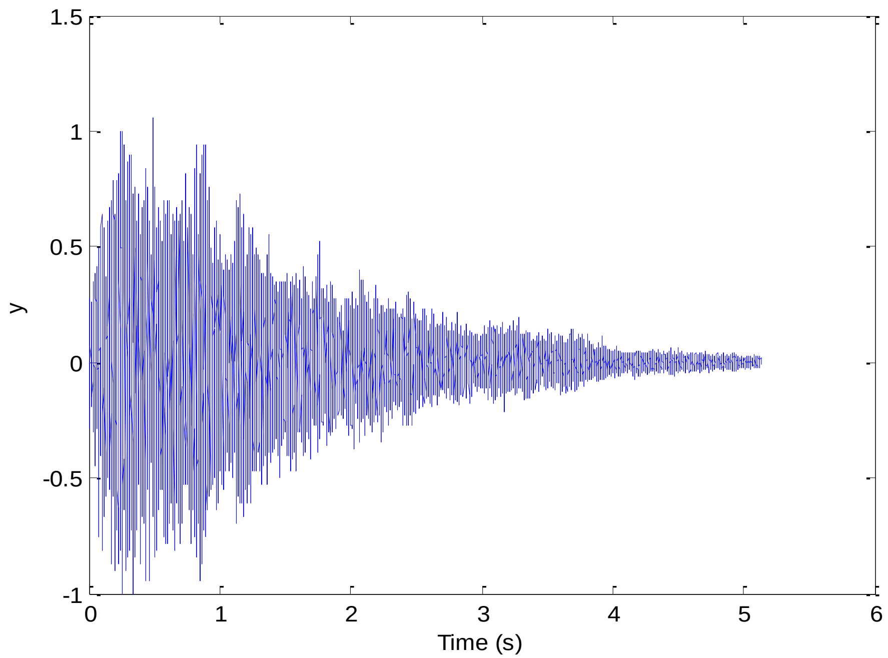




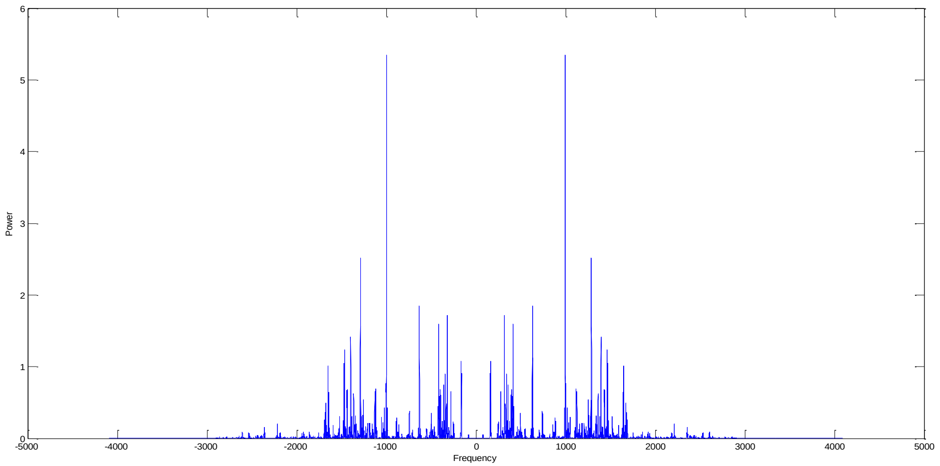




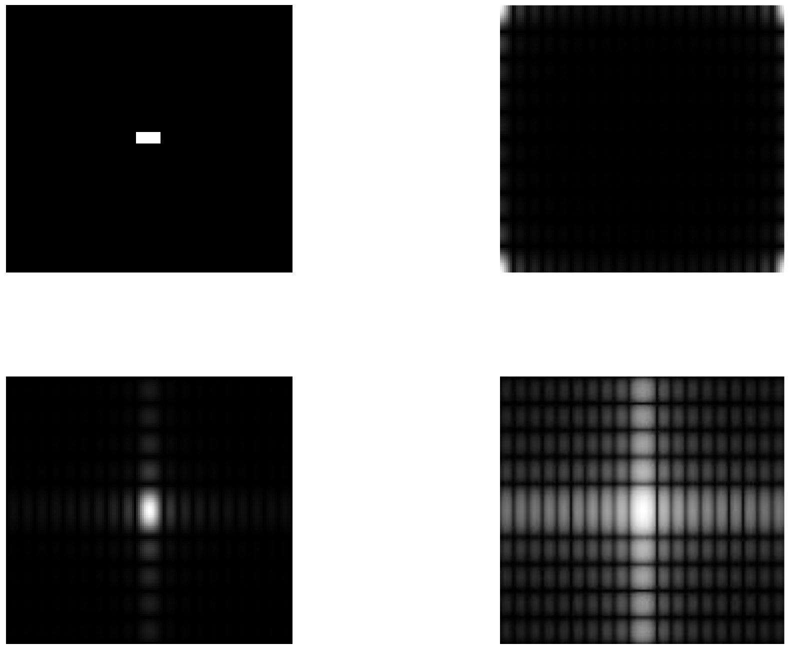
```
clear all
close all


load('gong') %load the variables for the 'gong' audio file, this loads the sample frequency and the sample values
Fs
t=0:1/Fs:length(y)/Fs-1/Fs; %time index
figure;plot(t,y);xlabel('Time (s)'),ylabel('y')
```




```
y1 = fft(y);
n = length(y1);          % number of samples
y2 = fftshift(y1);        % shift y values
f0 = (-n/2:n/2-1)*(Fs/n); % 0-centered frequency range
power0 = abs(y2).^2/n;    % 0-centered power

figure, plot(f0,power0)
xlabel('Frequency')
ylabel('Power')
```



```
clear all
close all

f = ones(10,20);
F = fft2(f, 500,500);
f1 = zeros(500,500);
f1(240:260,230:270) = 1;
subplot(2,2,1);imshow(f1,[]);
S = abs(F);
subplot(2,2,2); imshow(S,[]);
Fc = fftshift(F);
S1 = abs(Fc);
subplot(2,2,3); imshow(S1,[]);
S2 = log(1+S1);
subplot(2,2,4);imshow(S2,[]);
```



FFT for Spectral Analysis

First create some data. Consider data sampled at 1000 Hz. Start by forming a time axis for our data, running from t=0 until t=.25 in steps of 1 millisecond. Then form a signal, x, containing sine waves at 50 Hz and 120 Hz.

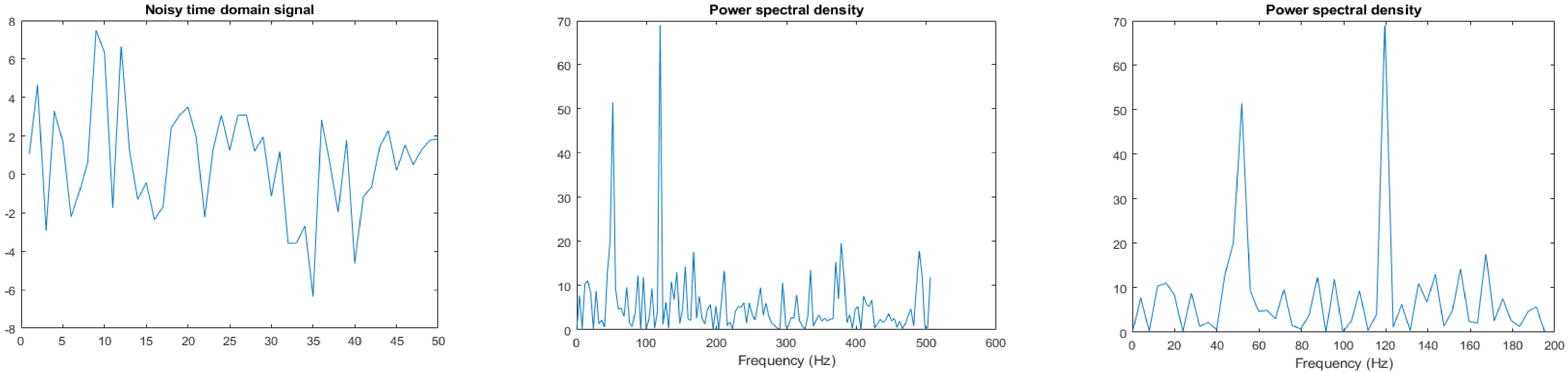
- t = 0:.001:.25;
- x = sin(2*pi*50*t) + sin(2*pi*120*t);

Add some random noise with a standard deviation of 2 to produce a noisy signal y. Take a look at this noisy signal y by plotting it.

- y = x + 2*randn(size(t));
- plot(y(1:50))
- title('Noisy time domain signal')
- Y = fft(y,251);
- Pyy = Y.*conj(Y)/251;
- f = 1000/251*(0:127);
- plot(f,Pyy(1:128))
- title('Power spectral density')
- xlabel('Frequency (Hz)')

Zoom in and plot only up to 200 Hz. Notice the peaks at 50 Hz and 120 Hz. These are the frequencies of the original signal.

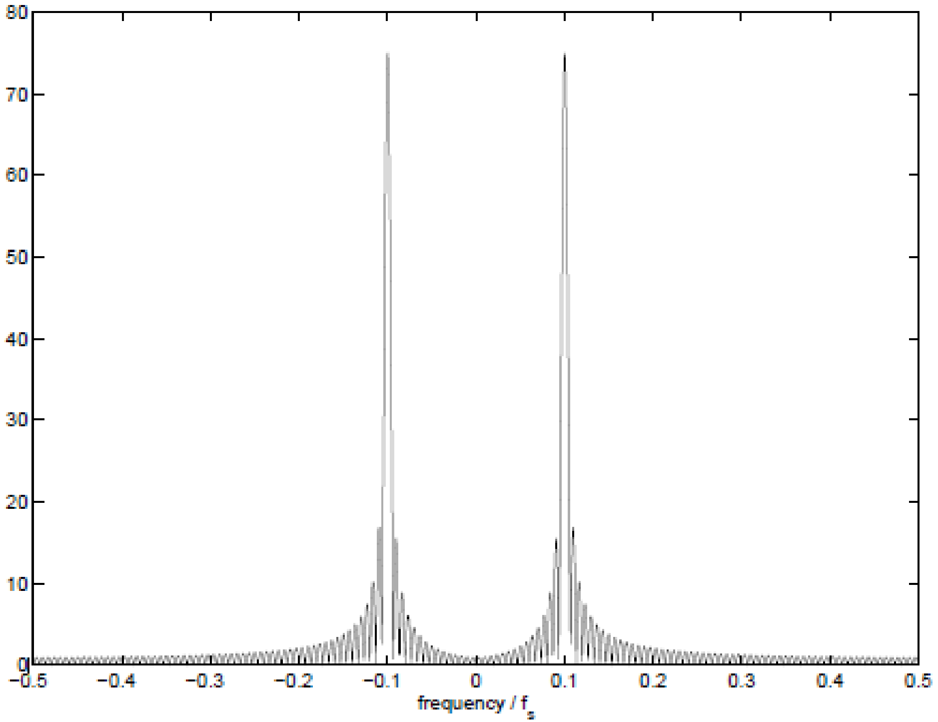
- plot(f(1:50),Pyy(1:50))
- title('Power spectral density')
- xlabel('Frequency (Hz)')



Noisy time domain signal
Power spectral density
Frequency (Hz)
Power spectral density
Frequency (Hz)

Approximate Spectrum of a Sinusoid with the FFT

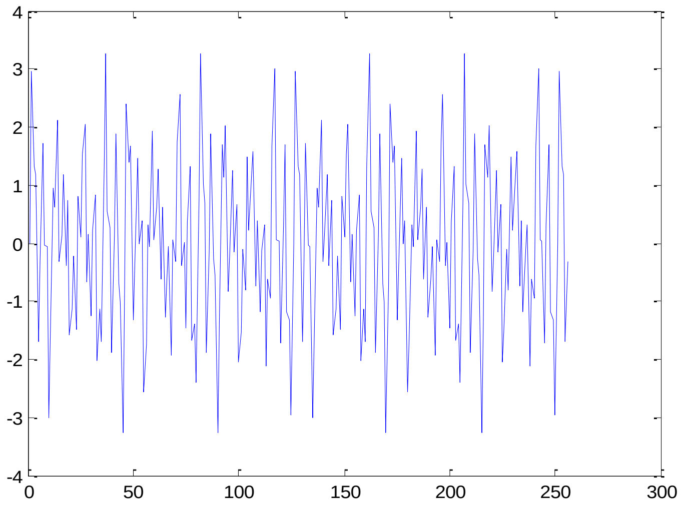
```
n = [0:149];
x1 = cos(2*pi*n/10);

N = 2048;

X = abs(fft(x1,N));
X = fftshift(X);

F = [-N/2:N/2-1]/N;

plot(F,X),
xlabel('frequency / f_s')
```



```
clear all
close all

f = imread('Jenna.jpg');
subplot(1,2,1), imshow(f);
f = double(f);
F = fft2(f);
Fc = fftshift(F);
S = log(1+abs(Fc));
subplot(1,2,2),imshow(S,[]);
```

Örnek

```
clear all
close all



% Signal parameters:
f = [ 440 880 1000 2000 ];      % frequencies
M = 256;                        % signal length
Fs = 5000;                      % sampling rate

% Generate a signal by adding up sinusoids:
x = zeros(1,M); % pre-allocate 'accumulator'
n = 0:(M-1);    % discrete-time grid
for fk = f;
   x = x + sin(2*pi*n*fk/Fs);
              t=n/Fs);
end

figure, plot(t,x)

ya=fft(x,1024);
```



2D Fourier Transform

- The 2D Fourier Transform equation is very similar to the 1D

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-i2\pi(\frac{u\,x}{M} + \frac{v\,y}{N})}$$

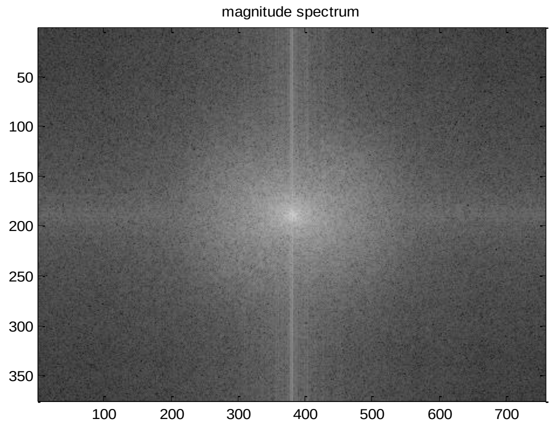
$$f(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} F(x, y) e^{i2\pi(\frac{u\,x}{M} + \frac{v\,y}{N})}$$

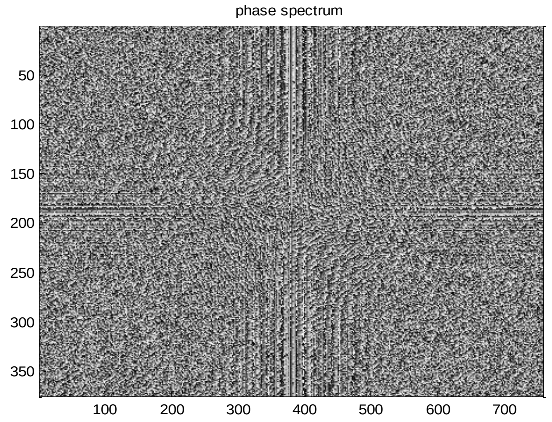
- The 1/MN term can be applied to either function (but not both) and is used for normalization

- In MATLAB the call for a 2D Fourier transform is

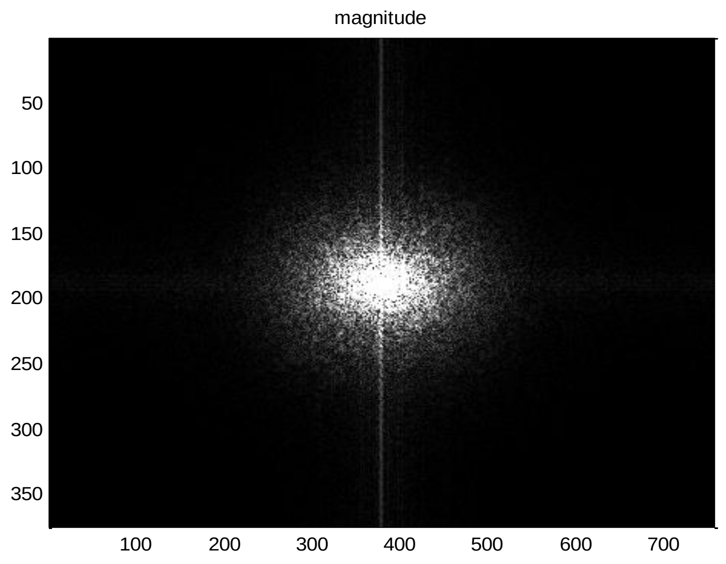
```
>>F = fft2(f)
>>f = ifft2(F)
```

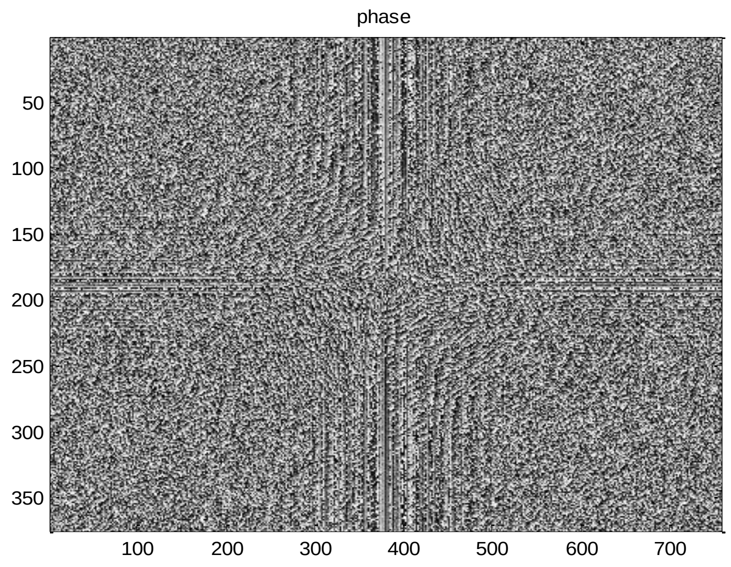
```
clear all
close all

%Create the Spacial Filtered Image
f = imread('Apricot.png');
whos f
size(f)
class(f)
[M, N] = size(f)
figure, imshow(f)

I = rgb2gray(f)
figure, imshow(I)

F=fft2(I)

figure, imagesc(100*log(1+abs(fftshift(F)))); colormap(gray);
title('magnitude spectrum');

figure,imagesc(angle(fftshift(F)));  colormap(gray);
title('phase spectrum');
```



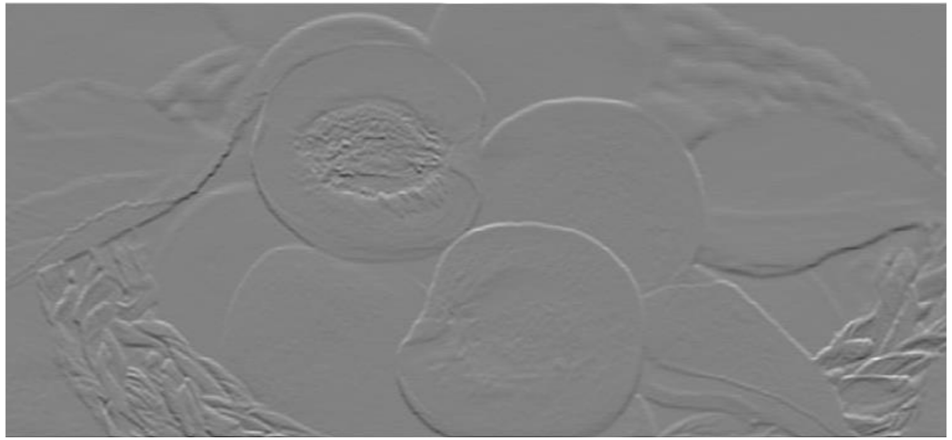


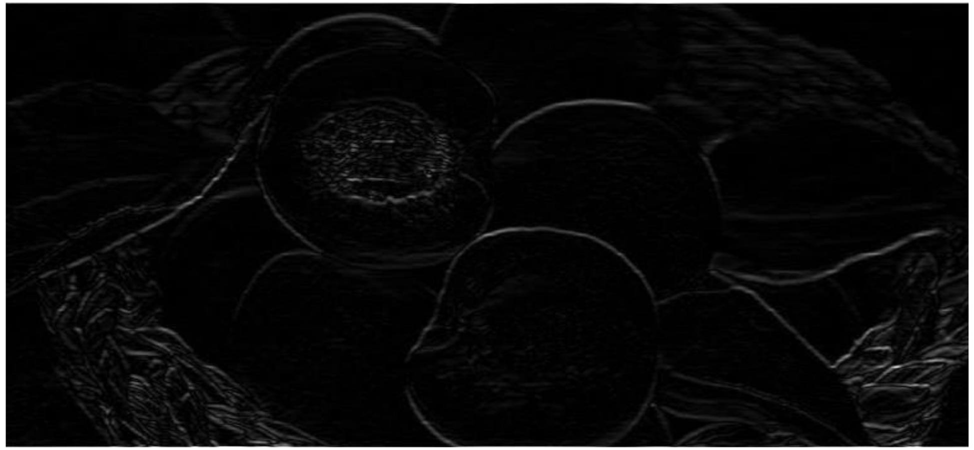
```
clear all
close all



im=imread('Apricot.png');
im=im(:,:,1);

imshow(im(:,:,1))

y=fft2(im);

clim=quantile(abs(y(:)),[.01 .99]);
figure
imagesc(fftshift(abs(y)),clim);colormap gray
title('magnitude');

clim=quantile(angle(y(:)),[.01 .99]);
figure
imagesc(fftshift(angle(y)),clim);colormap gray
title('phase')
```





```
clear all
close all

%Create the Spacial Filtered Image
f = imread('Apricot.png');
%Convert to grayscale
f1=rgb2gray(f);
figure,imshow(f1)

%Create the Spacial Filtered Image
h = fspecial('sobel');
sfi = imfilter(double(f1),h, 0, 'conv');
%Display results (show all values)
figure,imshow(sfi, []);


%The abs function gets correct magnitude
%when used on complex numbers
sfim = abs(sfi);
figure, imshow(sfim, []);
```

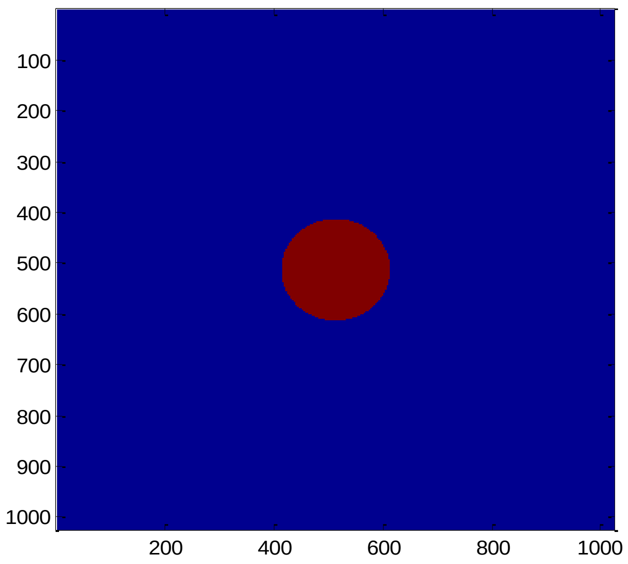


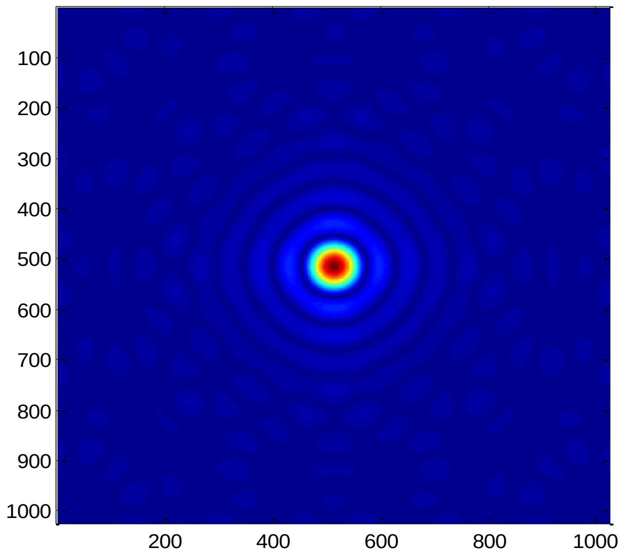


```
clear all
close all

%Create the Spacial Filtered Image
f = imread('Apricot.png');
whos f
size(f)
class(f)
[M, N] = size(f)
figure, imshow(f)
I = rgb2gray(f)


windowSize = 9;
kernel = ones(windowSize)/windowSize^2;
blurredImage = conv2(double(I), kernel, 'same');
figure, surf(blurredImage);
colormap(gray(256));
```

R değerini 1 ile 100 arasındaki değiştirerek değişimi görün

```
clear all
close all
n = 2^10;              % size of mask
M = zeros(n);
I = 1:n;
x = I-n/2;             % mask x-coordinates
y = n/2-I;             % mask y-coordinates
[X,Y] = meshgrid(x,y);    % create 2-D mask grid


R=10;                  % aperture radius
A = (X.^2 + Y.^2 <= R^2); % circular aperture of radius R
M(A) = 1;              % set mask elements inside aperture to 1
imagesc(M)             % plot mask
axis image
DP = fftshift(fft2(M));
figure, imagesc(abs(DP))
axis image
figure,imagesc(abs(log2(DP)))
```





Ses analizi

- clear all
- close all
- recObj = audiorecorder
- disp('Start speaking.')
- recordblocking(recObj, 15);
- disp('End of Recording.');

- play(recObj);

- y = getaudiodata(recObj);

- figure, plot(y);

- y1=fft(y);
- figure, plot(abs(y1))

- N=size(y1)
- for i=10000:110000
-     y1(i)=0;
- end

- figure, plot(abs(y1))

Kaynakça

- Fast Fourier Transform and MATLAB Implementation by Wanjun Huang for Dr. Duncan L. MacFarlane
- Borrowed from http://perso.ens-yon.fr/patrick.flandrin/emd.html , Gabriel Rilling and Patrick Flandrin

Usage Notes

- These slides were gathered from the presentations published on the internet. I would like to thank who prepared slides and documents.
- Also, these slides are made publicly available on the web for anyone to use
- If you choose to use them, I ask that you alert me of any mistakes which were made and allow me the option of incorporating such changes (with an acknowledgment) in my set of slides.

Sincerely,

Dr. Cahit Karakuş

cahitkarakus@esenyurt.edu.tr